2

Reliability Mathematics

2.1 INTRODUCTION

This chapter presents some basic mathematical concepts needed to understand subsequent chapters. Such topics as set theory, discrete and continuous random variables, probability distributions, hazard plotting, and differential equations are discussed briefly and provide an overview of the subject. The reader requiring in-depth knowledge of these concepts should consult references 1 and 5–7.

2.2 SET THEORY

Sets are normally represented by capital letters such as X, Y, Z. Elements are denoted by the lower case letters such as c, d, e.

If k is an element of set B, then it is denoted as: $k \in B$ and its negation is denoted as $k \notin B$. If X is a subset of set Y it is written as

$$X \subset Y \qquad \text{or} \qquad Y \supset X \tag{2.1}$$

The negation of the above is written as

$$X \not\subset Y \qquad \text{or} \qquad Y \not\supset X \tag{2.2}$$

If two sets are equal (suppose each set belongs to the other) they are expressed as

$$X = Y \tag{2.3}$$

The statement (2.3) is true if only

$$X \subset Y \qquad \text{and} \qquad Y \subset X \tag{2.4}$$

2.2.1 Union of Sets

The union of sets is denoted by the symbol $\cup$ or $+$. For example if $X + Y = Z$, it means that all the elements in set X or in set Y or in both sets

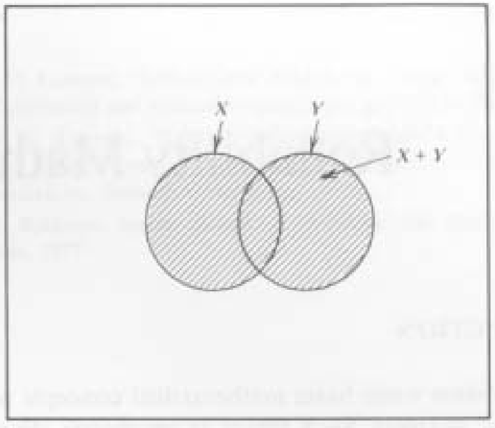

Figure 2.1 Venn diagram for the union of sets X, Y.

X and Y are contained in set Z. The statement

$$Z = X + Y \tag{2.5}$$

may also be written as $Z = X \cup Y$.

This case may be represented on the Venn diagram as shown in Figure 2.1.

2.2.2 *Intersection of Sets*

The intersection of sets is denoted by $\cap$ or dot $(\cdot)$. For example, if the intersection of sets or events C and D is represented by a third set, say T, then this set contains all elements which belong to both C and D. It is

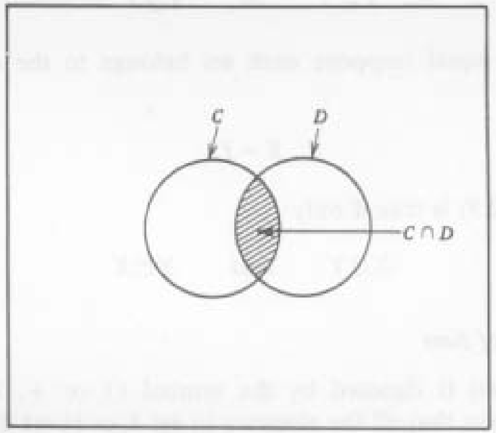

Figure 2.2 Venn diagram for intersection.

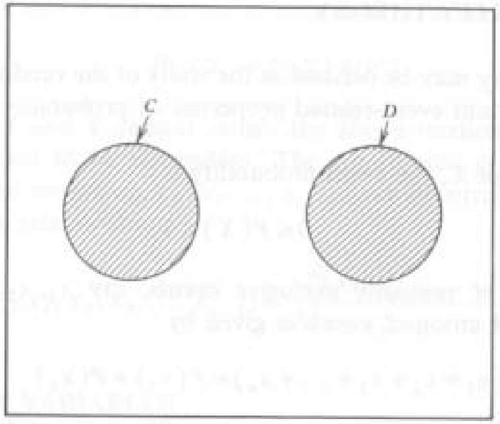

Figure 2.3 Venn diagram for disjoint sets C and D.

denoted as

$$T = C \cap D \quad \text{or} \quad T = C \cdot D \tag{2.6}$$

The above expression is shown on the Venn diagram in Figure 2.2.

If the intersection of sets C and D is zero then sets C and D are called mutually exclusive or disjoint sets. This may be represented on Venn diagram as shown in Figure 2.3.

2.2.3 *Basic Laws of Boolean Algebra*

Some laws of Boolean algebra are as follows:

1. Distributive laws

$$X(Y + Z) = (X \cdot Y) + (X \cdot Z) \tag{2.7}$$

$$X + (Y \cdot Z) = (X + Y) \cdot (X + Z) \tag{2.8}$$

2. Boolean identities

$$X + X = X \tag{2.9}$$

$$X \cdot X = X \tag{2.10}$$

3. Absorption laws

$$X + (X \cdot Y) = X \tag{2.11}$$

$$X(X \cdot Y) = X \cdot Y \tag{2.12}$$

2.3 PROBABILITY THEORY

Probability theory may be defined as the study of the random experiments. The most important event-related properties of probability are as follows:

1. For each event X, the event probability is

$$0 \leqslant P(X) \leqslant 1 \tag{2.13}$$

2. In the case of mutually exclusive events, say $x_1, x_2, x_3, \ldots, x_n$, the probability of union of events is given by

$$P(x_1+x_2+x_3+\ldots+x_n)=P(x_1)+P(x_2)+P(x_3)+\ldots+P(x_n) \tag{2.14}$$

3. The union of n events is given by

$$P(x_1+x_2+x_3+\ldots+x_n)=\{P(x_1)+P(x_2)+\ldots+P(x_n)\} - \{P(x_1x_2)+P(x_1x_3)+\ldots+P(x_jx_{ij\neq i})\}+\ldots +(-1)^{n-1}\{P(x_1x_2x_3\ldots x_n)\} \tag{2.15}$$

For example, in the case of two statistically independent events x_1 and x_2, the probability expressions becomes:

$$P(x_1+x_2)=P(x_1)+P(x_2)-P(x_1)P(x_2) \tag{2.16}$$

4. Probability of the sample space S is always equal to unity, that is,

$$P(S)=1 \tag{2.17}$$

The negation of the sample space S is written as $\bar{S}$. Thus

$$P(\bar{S})=0 \tag{2.18}$$

5. The n events intersection probability expression is as follows

$$P(x_1x_2x_3\ldots x_n)=P(x_1)P(x_2/x_1)\ldots P(x_n/x_1x_2\ldots x_{n-1}) \tag{2.19}$$

where $P(x_2/x_1)$ implies probability of x_2 given x_1.

If all the events are statistically independent, the above expression becomes

$$P(x_1x_2x_3\ldots x_n)=P(x_1)P(x_2)P(x_3)\ldots P(x_n) \tag{2.20}$$

6. The events X and Y are said to be independent, if and only if

$$P(XY)=P(X)P(Y) \tag{2.21}$$

If events X and Y cannot satisfy the above relationship, then these events are said to be dependent. The conditional probability of x_n, given that the events $x_1, x_2, x_3, \ldots, x_{n-1}$ have occurred is obtained by the following relationship:

$$P(x_n/x_1, x_2, x_3, \ldots, x_{n-1})=\frac{P(x_1, x_2, x_3, \ldots, x_n)}{P(x_1, x_2, x_3, \ldots, x_{n-1})} \tag{2.22}$$

2.4 RANDOM VARIABLES

Random variables may be discrete or continuous. Both discrete and continuous variables and the associated probability distributions are described in these sections.

2.4.1 Discrete Random Variables

If Y is a random variable on the sample space S along with a countably infinite set $Y(S)=\{y_1, y_2, y_2, \ldots\}$, then these random variables along with other finite sets are known as discrete random variables.

Density Function. For a single-dimension discrete random variable Y, the discrete probability function of the random variable Y is represented by $f(y_i)$ if the following conditions hold:

$$f(y_i) \geqslant 0 \quad \text{for all} \quad y_i \in R_y \text{ (range space)} \tag{2.23}$$

and

$$\sum_{\substack{\text{all}\\ y_i}} f(y_i)=1 \tag{2.24}$$

Cumulative Probability Distribution Function. The cumulative probability distribution function is defined as

$$F(y)=\sum_{y_i \leqslant y} f(y_i) \tag{2.25}$$

where $F(y)$ is the cumulative probability distribution function.

Furthermore, the area under the probability density function curve is always

$$0 \leqslant F(y) \leqslant 1 \tag{2.26}$$

Binomial Distribution. The binomial distribution is a frequently used distribution in reliability engineering. This is also known as the Bernoulli distribution. We are often concerned with the probabilities of outcome such as the total number of failures in a sequence of n trials. For this distribution, each trial has two possible outcomes, success and failure, where the probability of each trial remains constant.

The binomial probability function $f(x)$ is defined as

$$f(x)=\frac{n!}{x!(n-x)!}p^x q^{n-x}, \qquad x=0,1,2,\ldots,n \tag{2.27}$$

where x = the number of failures in n trials
p = the single trial probability of success
q = the single trial probability of failure

It is always true that the summation of probability of failure and success for each trial is always equal to unity (i.e., $p+q=1$).

The probability of x or less failures in n number of trials is known as the probability distribution function, $F(x)$, i.e.

$$F(x)=\sum_{i=0}^{x}\binom{n}{i}p^i q^{n-i} \tag{2.28}$$

where $\binom{n}{i}=n!/i!(n-i)!$.

Poisson Distribution. This distribution model is used in reliability studies when one is interested in the occurrence of a number of events that are of the same kind. Occurrence of each event is represented as a point on a time scale. In reliability engineering each event represents a failure. The Poisson density function is defined as

$$f(n)=\frac{(\lambda t)^n \exp(-\lambda t)}{n!}, \qquad n=0,1,2,\ldots \tag{2.29}$$

where t is the time and λ is the constant failure or arrival rate.

The cumulative distribution function F is given by

$$F=\sum_{i=0}^{n}\frac{(\lambda t)^i \exp(-\lambda t)}{i!} \tag{2.30}$$

Multinomial Distribution. This distribution is applicable to those cases where a system or device has more than two states. This is an extension of the binomial distribution which is only applicable to systems or devices with two states. The multinomial distribution probability function is defined as follows:

$$f(x_1,x_2,x_3,\ldots,x_n)=\frac{n!}{x_1!x_2!x_3!\ldots x_n!}P_1^{x_1}P_2^{x_2}P_3^{x_3}\ldots P_n^{x_n} \tag{2.31}$$

$$\text{for} \quad \left.\begin{array}{l}\sum_{i=1}^{n}P_i=1\\ \sum_{i=1}^{n}x_i=n\end{array}\right\} \quad 0<P_i<1$$

2.4.2 Continuous Random Variables

A real-valued function defined over a sample space S is called a continuous random variable. In the case of the continuous random variable, the probability density function is defined as

$$f(t)=\frac{dF(t)}{dt} \tag{2.32}$$

where

$$F(t)=\int_{-\infty}^{t}f(x)\,dx \tag{2.33}$$

and

$$F(\infty)=1$$

$F(t)$ is called the distribution function of a continuous random variable t. The probability distributions of the continuous random variable are as follows:

Uniform Distribution. This is a continuous distribution whose probability density $f(t)$ and distribution functions $F(t)$, respectively, are defined as follows:

$$f(t)=\frac{1}{\alpha-\theta} \qquad \theta<t<\alpha \tag{2.34}$$

otherwise

$$f(t)=0$$

and

$$F(t)=\begin{cases} 1 & t\geq\alpha \\ 0 & t<\theta \\ \dfrac{t-\theta}{\alpha-\theta} & \theta<t<\alpha \end{cases} \tag{2.35}$$

Exponential Distribution. This is a widely used distribution in reliability engineering [2]. It is one of the simplest distributions to perform reliability analysis. The exponential probability density function $f(t)$ is defined as follows:

$$f(t)=\lambda e^{-\lambda t} \qquad t\geq 0 \qquad \lambda>0 \tag{2.36}$$

where λ is a constant failure rate and t is time.

The cumulative distribution function $F(t)$ is given by

$$F(t)=1-e^{-\lambda t} \tag{2.37}$$

Weibull Distribution. This distribution is due to Weibull [8]. This distribution can represent many different physical phenomena. Weibull distribution is a three parameters distribution whose probability density function is defined as follows:

$$f(t)=\frac{b}{n}(t-\alpha)^{b-1}e^{-((t-\alpha)^b/n)} \qquad \text{for} \quad t>\alpha \qquad b,n,\alpha>0 \tag{2.38}$$

where b, n, and α are shape, scale, and location parameters, respectively.

The distribution function is given by

$$F(t)=1-e^{-((t-\alpha)^b/n)} \qquad \text{for} \quad t>\alpha \qquad n,b>0 \qquad \alpha\geq 0 \tag{2.39}$$

Rayleigh Distribution. This distribution has its applications in the theory of sound and reliability engineering. The Rayleigh distribution is a special case of the Weibull distribution ($b=2$, $\alpha=0$). Therefore, the probability density and distribution functions may be directly obtained from (2.38) and (2.39), respectively, as follows:

$$f(t)=\frac{2}{n}te^{-t^2/n} \qquad t\geq 0 \qquad n>0 \tag{2.40}$$

and

$$F(t)=1-e^{-t^2/n} \tag{2.41}$$

Gamma Distribution. This distribution is an extension of the exponential distribution. Some of its applications are found in life test problems.

Probability density and distribution functions are

$$f(t)=\frac{\lambda(\lambda t)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda t} \qquad t\geq 0 \qquad \lambda,\alpha>0 \tag{2.42}$$

and

$$F(t)=1-\sum_{i=0}^{\alpha-1}\frac{e^{-\lambda t}(\lambda t)^i}{i!} \qquad t\geq 0 \qquad \lambda,\alpha>0 \tag{2.43}$$

In the case of $\alpha=1$, this distribution reduces to exponential form.

Extreme Value Distribution. It is a good representative of the failure behavior of mechanical components. Probability density and distribution function of the extreme value distribution are as follows:

$$f(t)=e^te^{-e^t} \qquad -\infty<t<\infty \tag{2.44}$$

and

$$F(t)=1-e^{-e^t} \qquad -\infty<t<\infty \tag{2.45}$$

Normal Distribution (Gaussian). This is a two-parameter distribution, which also has its applications in the reliability field. Its probability density function is defined as follows:

$$f(t)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-1/2\frac{(t-\mu)^2}{\sigma}} \qquad -\infty<t<\infty \qquad \sigma>0 \qquad -\infty<\mu<\infty \tag{2.46}$$

The cumulative distribution function is

$$F(t)=\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{t}e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}dt \tag{2.47}$$

The numerical values of the cumulative function (2.47) may be obtained from the standard tables.

Log Normal Distribution. This is another distribution often used to represent the repair times of failed equipment. The probability density and distribution functions are

$$f(t)=\frac{1}{(t-\alpha)\sqrt{2\pi}\,\sigma}e^{-(\log_e(t-\alpha)-\mu)^2/2\sigma^2} \qquad \text{for} \quad t>\alpha>0 \qquad \sigma>0 \tag{2.48}$$

and

$$F(t)=\frac{1}{\sqrt{2\pi}\,\sigma}\int_0^t \frac{1}{x}e^{-(\ln x-\mu)^2/2\sigma^2}\,dx \qquad \text{for} \quad t>0 \qquad (2.49)$$

Beta Distribution. The beta distribution is a two-parameter distribution finding some uses in reliability engineering. The probability density function of this distribution is defined as follows:

$$f(t)=\frac{(\gamma+\beta+1)!}{\gamma!\beta!}t^{\gamma}(1-t)^{\beta}$$

$$\text{for} \quad 0<t<1 \qquad \gamma>-1 \qquad \beta>-1 \qquad (2.50)$$

The cumulative distribution function is given by

$$F(t)=\int_0^t \frac{(\gamma+\beta+1)}{\gamma!\beta!}y^{\gamma}(1-y)^{\beta}\,dy \qquad \text{for} \quad 0<t<1 \qquad (2.51)$$

The General Distribution (Hazard-Rate Model). This section presents a general distribution [3] which might be useful to represent failure behavior of items that are not adequately represented by the existing failure distributions.

The hazard rate $\lambda(t)$ and reliability function $R(t)$ are defined by

$$\lambda(t)=k\lambda ct^{c-1}+(1-k)bt^{b-1}\beta e^{\beta t^b}$$

$$\text{for} \quad b,c,\beta,\lambda>0 \qquad 0\leqslant k\leqslant 1 \qquad t\geqslant 0 \qquad (2.52)$$

and

$$R(t)=\exp\left[-k\lambda t^c-(1-k)(e^{\beta t^b}-1)\right] \qquad (2.53)$$

where b, c = shape parameters
β, λ = scale parameters
t = time

In special cases, the above distribution becomes

$c=1, b=1$	Makeham distribution
$k=0, b=1$	extreme value
$k=1$	Weibull
$c=0.5, b=1$	bathtub curve

The Hazard Rate Model Distribution. The hazard rate function $\lambda(t)$ [4] of this model is defined as follows:

$$\lambda(t)=k\lambda\tanh\lambda t+(1-k)bt^{b-1}\beta e^{-\beta t^b}$$

$$\text{for} \quad b,\beta,\lambda>0 \qquad 0\leqslant k\leqslant 1 \qquad t\geqslant 0 \qquad (2.54)$$

where b = the shape parameter
β, λ = the scale parameters

The reliability function is given by

$$R(t)=\exp\{-k\ln\cosh\lambda t+(1-k)(e^{-\beta t^b}-1)\} \qquad (2.55)$$

Figure 2.4 shows some selective curves ($\beta=\lambda=1$) for the hazard rate function expressed in (2.54).

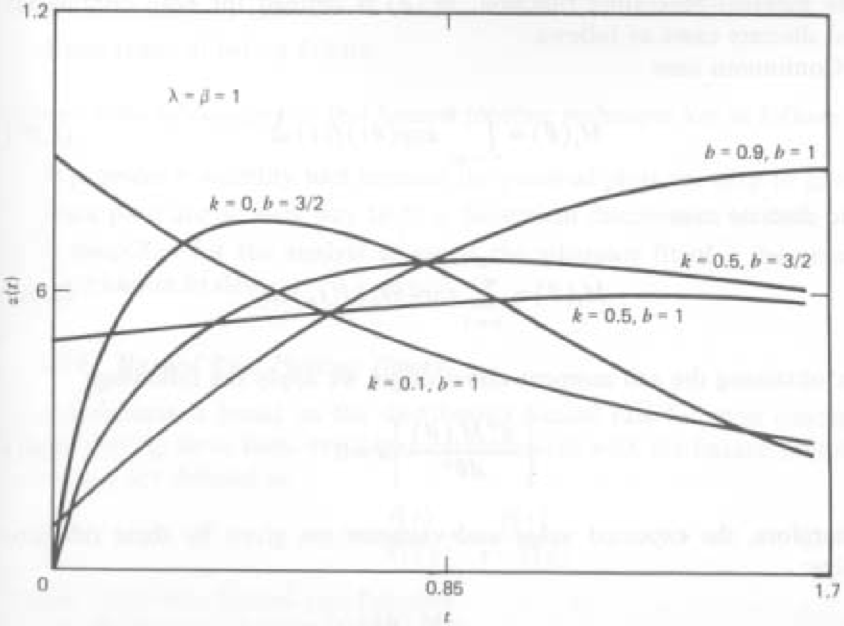

Figure 2.4 Hazard rate function plot.

2.5 EXPECTED VALUE AND VARIANCE OF THE RANDOM VARIABLES

The expected value, $E(x)$, of a continuous random variable is defined as

$$E(x)=\int_{-\infty}^{\infty} xf(x)\,dx \tag{2.56}$$

Similarly in the case of a discrete random variable x, the expected value, $E(x)$, is given by

$$E(x)=\sum_{i=1}^{k} x_i f(x_i) \tag{2.57}$$

where k are the discrete values of the random variable x.

The variance $\sigma^2(x)$ of a random variable x is defined by

$$\sigma^2(x)=E(x^2)-\{E(x)\}^2 \tag{2.58}$$

2.6 MOMENT-GENERATING FUNCTION

The moment-generating function, $M_t(\theta)$ is defined for both continuous and discrete cases as follows:

Continuous case

$$M_t(\theta)=\int_{-\infty}^{+\infty} \exp(\theta t)f(t)\,dt \tag{2.59}$$

and discrete case

$$M_t(\theta)=\sum_{k=1}^{n} \exp(\theta t_k)f(t_k)\,dt \tag{2.60}$$

for obtaining the nth moment about origin we apply the following:

$$\left[\frac{d^nM_t(\theta)}{d\theta^n}\right]\theta=0 \tag{2.61}$$

Therefore, the expected value and variance are given by these relationships:

$$E(t)=\left.\frac{dM_t(\theta)}{d\theta}\right|_{\theta=0} \tag{2.62}$$

and

$$E(t^2)=\left.\frac{d^2M_t(\theta)}{d\theta^2}\right|_{\theta=0} \tag{2.63}$$

2.7 HAZARD PLOTTING FOR INCOMPLETE FAILURE DATA

This is a graphical data analysis technique [7] to establish failure distributions for units with incomplete failure data. Failure data are complete failure data if the failure times for all units in a sample are contained. In contrast, the failure data are called incomplete failure data if a sample contains both the failure times of failed units and running times of unfailed units. The unfailed units running times are called censoring times.

In addition, if in a sample all the unfailed units under observation have different censoring times, then the failure data are called multiply censored. Furthermore, if the unfailed units in a sample have the same censoring time and in addition the censoring time is greater than the failure times, then the failure data are called singly censored. This type of data results when a sample of items undergo life testing and termination of testing before all units fail, whereas the multiply censored data result from any of the following:

1. From the operating units.
2. Some extraneous causes.
3. Units removal before failure.

Some of the advantages of this hazard plotting technique are as follows:

1. It provides a visibility tool because the pictorial plots are easy to grasp.
2. Data plots are an easy way to fit a theoretical distribution to data.
3. It simplifies for the analyst to assess the adequate fit of a theoretical distribution to data.

2.7.1 *Hazard Rate Plotting Theory*

This technique is based on the distribution hazard rate function concept. The following three basic relationships associated with the hazard plotting technique are defined as

$$z(t)=\frac{f(t)}{R(t)}=\frac{f(t)}{1-F(t)} \tag{2.64}$$

where $z(t)$ = the hazard rate function
$R(t)$ = the reliability function
$F(t)$ = the failure distribution function

The cumulative hazard, $z_c(t)$, is given by

$$z_c(t)=\int_{-\infty}^{t} z(t)\,dt=-[\ln\{1-F(t)\}] \tag{2.65}$$

The cumulative distribution function $F(t)$ is defined as

$$F(t)=1-e^{-z_c(t)} \tag{2.66}$$

The above relationship is very useful to determine hazard function properties.

2.7.2 *Hazard Plotting for the Weibull Distribution*

This example is presented for the Weibull distribution. However, interested readers should consult reference 7 for other distributions as well as for a detailed presentation of this approach. Here the theory behind the Weibull hazard plotting is briefly described.

The Weibull hazard, $z(t)$, and the density function, $f(t)$, are defined as follows:

$$f(t)=\frac{\alpha t^{\alpha-1}}{\beta^{\alpha}}e^{-(t/\beta)^{\alpha}} \qquad \alpha,\beta\geqslant 0 \qquad t\geqslant 0 \tag{2.67}$$

and

$$z(t)=\frac{\alpha t^{\alpha-1}}{\beta^{\alpha}} \tag{2.68}$$

Both cumulative distribution and hazard functions are obtained by integrating expressions (2.67) and (2.68) over the time interval $[0,t]$ as follows:

$$F(t)=1-e^{-(t/\beta)^{\alpha}} \tag{2.69}$$

and

$$z_c(t)=(t/\beta)^{\alpha} \tag{2.70}$$

By taking the $\log_e$ of (2.70) we get

$$\ln(t)=\alpha^{-1}\ln(z_c)+\ln(\beta) \tag{2.71}$$

The above equation indicated that the left-hand side of this expression is the linear function of $\ln(z_c)$, which indicates that the log-log graph paper is the Weibull hazard paper. Therefore, parameters α and β can be estimated graphically by using the log-log paper.

The shape parameter, α, is estimated from the fact that $1/\alpha$ is the slope of the straight line. At $z_c=1$, the value of the β is equal to time t, therefore, by using this relationship, the value of the scale parameter β can be estimated.

2.8 LAPLACE TRANSFORMS

Some of these transforms are used in this book to solve systems of linear differential equations with constant coefficients. Furthermore, these transforms are applied in conjunction with other differential equation techniques to solve simpler type of partial differential equations. The basic definition of the Laplace transform $f(s)$, of a function $f(t)$ is as follows:

$$f(s)=\mathcal{L}\{f(t)\}=\int_0^{\infty}e^{-st}f(t)\,dt \tag{2.72}$$

where s = the Laplace transform variable
t = the time variable

Example 1. Find the Laplace transform of the function $f(t)=t$, that is,

$$f(s)=\int_0^{\infty}e^{-st}t\,dt=\left[\frac{e^{-st}}{-s}\left(t+\frac{1}{s}\right)\right]_0^{\infty}$$

$$=\frac{1}{s^2} \quad \text{for } s>0 \tag{2.73}$$

Example 2. If $f(t)=e^{at}$, the Laplace transform of this exponential function becomes

$$f(s)=\int_0^{\infty}e^{-st}e^{at}\,dt=\int_0^{\infty}e^{(a-s)t}$$

$$=\left[-\frac{1}{(s-a)}e^{-(s-a)t}\right]_0^{\infty}$$

$$=\frac{1}{s-a} \quad \text{for } s>a \tag{2.74}$$

2.8.1 *Laplace Theorem of Derivatives*

If $\mathcal{L}\{f(t)\}=f(s)$, then

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\}=sf(s)-f(0) \tag{2.75}$$

2.8.2 *Laplace Transform Initial-Value Theorem*

If the following limits exist, then the Abel's theorem is

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sf(s) \tag{2.76}$$

2.8.3 *Laplace Transform Final-Value Theorem*

Provided the following limits exist, then the final-value theorem may be stated as:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s) \tag{2.77}$$

Laplace Transform Table

$f(t)$	$f(s)$	
t	$1/s$	
$\frac{df(t)}{dt}$	$sf(s)-f(0)$	
$\frac{d^2f(t)}{dt^2}$	$s^2f(s)-sf(0)-f'(0)$	
1	$\frac{1}{s}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
t^k	$\frac{k!}{s^{k+1}}$	$s>0$

2.9 PARTIAL FRACTION TECHNIQUE

This is used when finding inverse Laplace transforms of a rational function such as $G(s)/Q(s)$, where $G(s)$ and $Q(s)$ are polynomials, and the degree of $G(s)$ is less than that of $Q(s)$. Therefore, the ratio of $G(s)/Q(s)$ may be written as the sum of rational functions or partial fractions in the following forms:

$$\frac{A}{(\alpha s+\beta)^n},\ \frac{Bs+C}{(\alpha s^2+\beta s+C)^n} \qquad n=1,2,3,\ldots$$

Heaviside Theorem. This is used to obtain partial fractions and inverse of a rational function, $G(s)/Q(s)$.

The inverse of $G(s)/Q(s)$ may be written as:

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{Q(s)}\right\} = \sum_{i=1}^{k} \frac{G(\beta_i)}{Q'(\beta_i)} e^{\beta_i t} \tag{2.78}$$

where the prime represents derivative with respect to s, β_i represents ith zero and k denotes total number of distinct zeros of $Q(s)$.

Example 3. Suppose

$$\frac{G(s)}{Q(s)} = \frac{s+2}{(s-4)(s-6)}$$

find the inverse Laplace transform. Hence,

$$G(s)=s+2 \qquad Q(s)=s^2-10s+24 \qquad Q'(s)=2s-10$$

$$\beta_1=4 \qquad \beta_2=6 \qquad k=2$$

Therefore,

$$\frac{G(4)}{Q'(4)}e^{4t} + \frac{G(6)}{Q'(6)}e^{6t} = 4e^{6t}-3e^{4t} \tag{2.79}$$

2.10 DIFFERENTIAL EQUATIONS

The single-independent-variable linear first-order differential equations in the reliability study are mainly associated with the Markov technique. In this section we discuss how to solve such equations using integration techniques.

The first-order first-degree linear differential equation may be written in the following form

$$\frac{dP}{dt} + PG(t) = Q(t) \tag{2.80}$$

Since

$$\frac{d}{dt}\left(Pe^{\int G(t)\,dt}\right) = \frac{dP}{dt}e^{\int G(t)\,dt} + PG(t)e^{\int G(t)\,dt}$$

$$= e^{\int G(t)\,dt}\left[\frac{dP}{dt} + PG(t)\right] \tag{2.81}$$

The above expression shows that $e^{\int G(t)\,dt}$ is an integrating factor of the differential equation (2.80).

The primitive of differential equation (2.80) may be written as

$$Pe^{\int G(t)\,dt} = \int Q(t)e^{\int G(t)\,dt}\,dt + c \tag{2.82}$$

where c is a constant.

Example 4. Obtain the solution equation for the following differential equation:

$$\frac{dP}{dt}+6P=8 \tag{2.83}$$

Hence,

$$e^{\int G(t)\,dt}=e^{\int 6\,dt}=e^{6t} \tag{2.84}$$

By substituting (2.84) into (2.82), we get

$$Pe^{6t}=8\int e^{6t}\,dt+c$$

$$P=\tfrac{8}{6}+ce^{-6t} \tag{2.85}$$

For given initial conditions; at $t=0$, $P=1$; the following value for the constant c is obtained from (2.85).

$$c=-\tfrac{1}{3}$$

$$P=\tfrac{4}{3}-\tfrac{1}{3}e^{-6t} \tag{2.86}$$

2.10.1 Differential Equation

Solution with Laplace Transform Technique. Solving the same differential equation

$$\frac{dP}{dt}+6P=8 \tag{2.87}$$

with the Laplace transform method for same initial conditions we get

$$sP-1+6P=\frac{8}{s}$$

$$P=\frac{8}{s(s+6)}+\frac{1}{s+6} \tag{2.88}$$

The inverse Laplace transform of the above equation is

$$P=\tfrac{4}{3}-\tfrac{1}{3}e^{-6t} \tag{2.89}$$

This shows that solution (2.86) is same as the solution to (2.89).

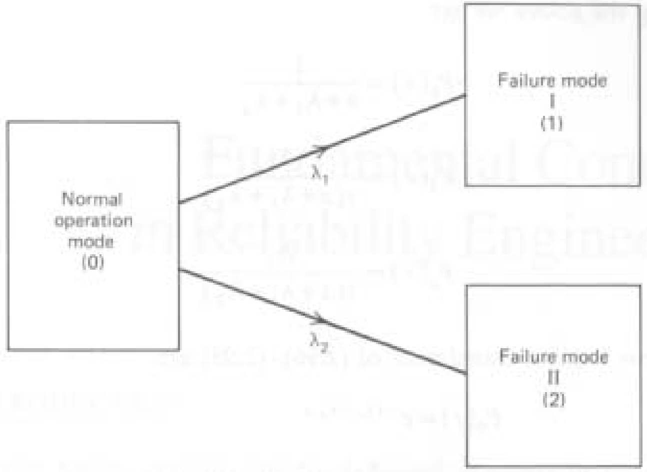

Figure 2.5 State space diagram.

2.10.2 System of Linear First Order Differential Equations

The following system of linear first-order differential equations with constant coefficients are associated with the transition diagram* of Figure 2.5. This transition diagram represents a three-state device, for example, a fluid flow valve, electronic diode, an electrical switch, etc.

$$\frac{dP_0(t)}{dt}+(\lambda_1+\lambda_2)P_0(t)=0 \tag{2.90}$$

$$\frac{dP_1(t)}{dt}-\lambda_1P_0(t)=0 \tag{2.91}$$

$$\frac{dP_2(t)}{dt}-\lambda_2P_0(t)=0 \tag{2.92}$$

At $t=0$, $P_0(t)=1$, and other probabilities are zero.

The Laplace transforms of differential equations (2.90)–(2.92) are

$$(s+\lambda_1+\lambda_2)P_0(s) \quad 0P_1(s) \quad 0P_2(s)=P_0(0) \tag{2.93}$$

$$-\lambda_1P_0(s) \quad sP_1(s) \quad 0P_2(s)=P_1(0) \tag{2.94}$$

$$-\lambda_2P_0(s) \quad 0P_1(s) \quad P_2(s)=P_2(0) \tag{2.95}$$

$$\begin{bmatrix} s+\lambda_1+\lambda_2 & 0 & 0 \\ -\lambda_1 & s & 0 \\ -\lambda_2 & 0 & s \end{bmatrix} \begin{bmatrix} P_0(s) \\ P_1(s) \\ P_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

*Markov technique is discussed in Chapter 3.

By solving the above we get

$$P_0(s)=\frac{1}{s+\lambda_1+\lambda_2} \tag{2.96}$$

$$P_1(s)=\frac{\lambda_1}{s(s+\lambda_1+\lambda_2)} \tag{2.97}$$

$$P_2(s)=\frac{\lambda_2}{s(s+\lambda_1+\lambda_2)} \tag{2.98}$$

The inverse Laplace transforms of (2.96)–(2.98) are

$$P_0(t)=e^{-(\lambda_1+\lambda_2)t} \tag{2.99}$$

$$P_1(t)=\frac{\lambda_1}{\lambda_1+\lambda_2}(1-e^{-(\lambda_1+\lambda_2)t}) \tag{2.100}$$

$$P_2(t)=\frac{\lambda_2}{\lambda_1+\lambda_2}(1-e^{-(\lambda_1+\lambda_2)t}) \tag{2.101}$$

REFERENCES

1. Bean, H. S., *Differential Equations*, Addison-Wesley, Reading, MA, 1962.
2. Davis, D. J., "An Analysis of Some Failure Data," *J. Amer. Stat. Assoc.*, 113–150 (June 1952).
3. Dhillon, B. S., "A Hazard Rate Model," *IEEE Trans. Reliab.*, **R-29**, 150 pp. (1979).
4. Dhillon, B. S., "New Hazard Rate Functions," *Microelectron. Reliab.* (1978).
5. Drake, A., *Fundamental of Applied Probability Theory*, McGraw-Hill, New York, 1967.
6. Feller, W., *An Introduction to Probability Theory and Its Applications*, Vol. I, Wiley, New York, 1957.
7. Nelson, W., "Hazard Plotting for Incomplete Failure Data," *J. Qual. Tech.*, 1 (1) (Jan. 1969).
8. Weibull, W., "A Statistical Distribution Function of Wide Applicability," *J. Appl. Mech.*, **18**, 293–297 (March 1951).