# 8

# Three-State Device Systems

## 8.1 INTRODUCTION

A three-state device operates satisfactorily in its normal mode but can fail in either of the two other modes. Typical examples of such a device are a fluid flow valve and an electronic diode. Closed (shorted) and open failure modes pertain to such devices.

Redundancy can generally be used to increase the reliability of a system without any change in the reliability of the individual devices that form the system. However, in the case of a system containing three-state devices, redundancy may either increase or decrease the system reliability. This depends upon the dominant mode of component failure, configuration of the system and the number of redundant components.

An electronic diode and a fluid flow valve are typical examples of three-state devices. Either of these components may fail catastrophically in either the open or closed (shorted) mode. A given three-state device will then have a probability of failure in the open-mode and a probability of failure in the closed or shorted mode. Because a three-state device cannot fail simultaneously in both the open and closed (shorted) modes, the failures are mutually exclusive events. The failure of any one such device is considered independent of all the others.

Three-state devices can be arranged in various redundant configurations such as series, parallel, series-parallel, parallel-series, and mixed arrangements. As these configurations become more complex, the analysis of networks becomes more cumbersome, and redundancy can result in decreased overall system reliability. This lower system reliability is due to the redundancy of the dominant adverse mode of failure.

## 8.2 LITERATURE REVIEW

Careful consideration of the reliability of three-state devices was presented by Moore and Shannon [27] and Creveling [7] in their 1956 papers on electrical and electronic devices. Creveling developed the reliability and failure equations for a diode quad arrangement, whereas Moore and Shannon developed formulas for several relay networks.

The year 1957 brought another development when Lipp [25] discussed the topology of switching elements versus reliability. The following year, Price [29] specifically dealt with the reliability of three-state devices in a parallel configuration and attempted to optimize the number of redundant components. In 1960, Barlow and Hunter [1–3] used calculus to optimize the reliability of series, parallel, series-parallel, and parallel-series networks. They also computed the number of components that maximize the expected system life for these first two types of systems assuming component life is exponentially distributed.

In 1962 Sorensen [35] applied the theory established by the previous researchers on three-state device networks to several electronic circuits. His primary approach was very similar to that of Creveling. In the same year, Cluley [6] published a paper on low-level redundancy as a means of improving the reliability of digital computers. Also in 1962 James et al. [23] reviewed the reliability problem and derived some systems reliability equations for redundant three-state device structures. In 1963, Blake [4] extended the work of Moore and Shannon [27] on networks of relay contacts by investigating the open and short circuit failures of hammock networks. Barlow et al. [3] extended their previous contribution to maximize the expected system life for components having exponential and uniform time to failure distributions.

In 1967 Kolesar [24] extended the work of the previous researchers when he optimized a series-parallel three-state device structure under constrained conditions. In 1970 Misra and Rao [26] developed a signal flow graph approach. During the following 2 years, only one of the four studies making reference to the subject appears to be important. Evans [19] gave a very brief introduction to three-state device reliabilities in his paper and Butler [5] made brief reference to it in his publication.

Since 1975 several contributions on the subject have been made by Dhillon [8–17, 30–34].

## 8.3 RELIABILITY ANALYSIS OF THREE-STATE DEVICE NETWORKS

The system reliability equations are developed for several configurations in this section. More detailed derivations are described in Appendix.

### 8.3.1 Series Structure

In a series configuration any one component failing in an open mode causes system failure, whereas all elements of the system must malfunction in a shorted mode for the system to fail. The system reliability is given by (8.1).

$$R_s=\prod_{i=1}^{n}(1-q_{oi})-\prod_{i=1}^{n}q_{si} \tag{8.1}$$



where $R_s$ = the series system reliability
$n$ = the number of nonidentical independent three-state components
$q_{oi}$ = the probability of open-mode failure of component $i$
$q_{si}$ = the probability of short-mode failure of component $i$

In the case of component constant open and short mode failure rates, the open and short mode failure probability equations become [8]

$$q_o(t)=\frac{\lambda_o}{\lambda_o+\lambda_s}\{1-e^{-(\lambda_o+\lambda_s)t}\} \tag{8.2}$$

and

$$q_s(t)=\frac{\lambda_s}{\lambda_o+\lambda_s}\{1-e^{-(\lambda_o+\lambda_s)t}\} \tag{8.3}$$

where $\lambda_o$ = the open-mode constant failure rate
$\lambda_s$ = the short-mode constant failure rate
$t$ = time

The derivation of (8.2) and (8.3) are shown in Section 8.5.2. To obtain (8.2) and (8.3) set $\mu_1=\mu_2=0$ in (8.57) and (8.58), respectively. By substituting expressions (8.2) and (8.3) in (8.1) we get:

$$R(t)=\prod_{i=1}^{n}\left[1-\frac{\lambda_{oi}}{\lambda_{oi}+\lambda_{si}}\{1-e^{-(\lambda_{oi}+\lambda_{si})t}\}\right]-\prod_{i=1}^{n}\frac{\lambda_{si}}{\lambda_{oi}+\lambda_{si}}\{1-e^{-(\lambda_{oi}+\lambda_{si})t}\} \tag{8.4}$$

*Short Failure Mode Probability.* The system short or closed failure mode probability, $Q_s$, is given by

$$Q_s=\prod_{i=1}^{n}q_{si} \tag{8.5}$$

*Open Failure Mode Probability.* Probability of open mode failure for a series system is given by

$$Q_o=1-\prod_{i=1}^{n}(1-q_{oi}) \tag{8.6}$$

Where $Q_o$ is the probability of open mode failure of series network.

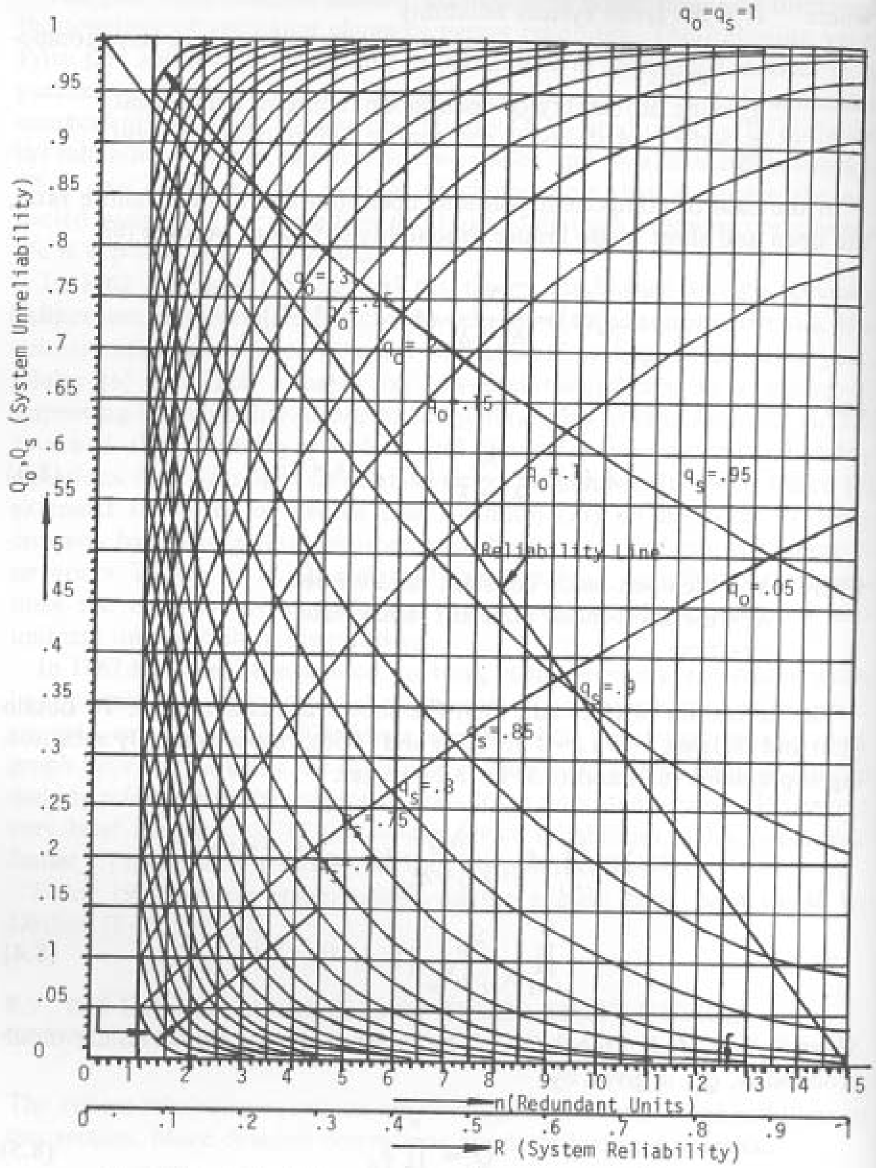


**Figure 8.1** An identical component series structure unreliability plot.

Plots of (8.5) and (8.6) are shown in Figure 8.1. This figure shows that the open mode failure probability increases as the number of redundant components in the series system increases.

*Example 1.* Consider two independent identical diodes connected in series. Open and short circuit failure probabilities are 0.2 and 0.1, respectively. It is required to find the system reliability of the two diodes for this simple arrangement.



In this case $n=2$, $q_s=0.1$ and $q_o=0.2$. Rewrite (8.1) for two identical diodes

$$R_s=(1-q_o)^2-q_s^2 \tag{8.7}$$

For given data,

$$R_s=(1-0.2)^2-(0.1)^2=0.63$$

### 8.3.2 Parallel Structure

For a parallel configuration, all the elements must fail in the open-mode or any one of the elements must be in a short-mode to cause the system to fail. The parallel network reliability is given by

$$R=\prod_{i=1}^{m}(1-q_{si})-\prod_{i=1}^{m}q_{oi} \tag{8.8}$$

where $m$ is the number of nonidentical independent elements.

The open and short failure mode probability plots are the same as shown in Figure 8.1. Because of duality, the short failure mode probability replaces the open failure probability and vice versa. The same duality concept applies to (8.1) and (8.8).

*Example 2.* Suppose the data of Example 1 is used for parallel configuration; evaluate the system reliability by using (8.8)

$$R=(1-q_s)^2-q_0^2=(1-0.1)^2-(0.2)^2=0.77$$

The parallel system reliability is 0.77.

### 8.3.3 Series-Parallel Network

This is a combination of series and parallel configurations. System reliability is given by (8.9) for $n$ identical independent units, each containing $m$ independent elements:

$$R=\left\{1-\prod_{i=1}^{m}q_{oi}\right\}^n-\left\{1-\prod_{i=1}^{m}(1-q_{si})\right\}^n \tag{8.9}$$

*Example 3.* Consider the reliability evaluation of series-parallel arrays of the identical fluid flow valves with $q_o=0.2$, $q_s=0.1$, $n=2$ and $m=4$.

For $n=2$ and $m=4$ (8.9) becomes

$$R=(1-q_o^4)^2-\{1-(1-q_s)^4\}^2 \tag{8.10}$$

For $q_s=0.1$, $q_o=0.2$, the system reliability

$$R=(1-0.2^4)^2-\{1-(1-0.1)^4\}^2=0.88$$

### 8.3.4 *Parallel-Series Structure*

This configuration is a dual of the series-parallel network. The system reliability equation for a configuration containing $m$ identical units and $n$ number of nonidentical series elements becomes

$$R=\left(1-\prod_{i=1}^{n} q_{si}\right)^m-\left[1-\prod_{i=1}^{n}(1-q_{oi})\right]^m \tag{8.11}$$

*Example 4.* Use the date given in Example 3 and evaluate the parallel-series network reliability. Therefore

$$\begin{aligned} R&=(1-q_s^2)^4-\{1-(1-q_o)^2\}^4 \\ &=(1-0.1^2)^4-\{1-(1-0.2)^2\}^4 \qquad (8.12) \\ &=0.9438 \end{aligned}$$

### 8.3.5 *Bridge Network*

This configuration is shown in Figure 8.2. The following bridge reliability equation, $R_b$ is taken from reference 25:

$$R_b=1-Q_{O1}-Q_{O2} \tag{8.13}$$

where $Q_{ok}$ is the network open failure mode probability, for $k=1$
$Q_{ok}$ is the network short (close) failure mode probability, for $k=2$
and

$$Q_{OK}=2\prod_{i=1}^{5}\Phi_i-\prod_{i=2}^{5}\Phi_i-\prod_{\substack{i=1\\ i\neq 2}}^{5}\Phi_i-\prod_{\substack{i=1\\ i\neq 3}}^{5}\Phi_i-\prod_{\substack{i=1\\ i\neq 4}}^{5}\Phi_i-\prod_{i=1}^{4}\Phi_i$$
$$+\prod_{\substack{i=1\\ i\neq 2,4}}^{5}\Phi_i+\prod_{i=2}^{4}\Phi_i+\prod_{\substack{i=1\\ i\neq 2,3}}^{4}\Phi_i+\prod_{\substack{i=2\\ i\neq 3,4}}^{5}\Phi_i \tag{8.14}$$



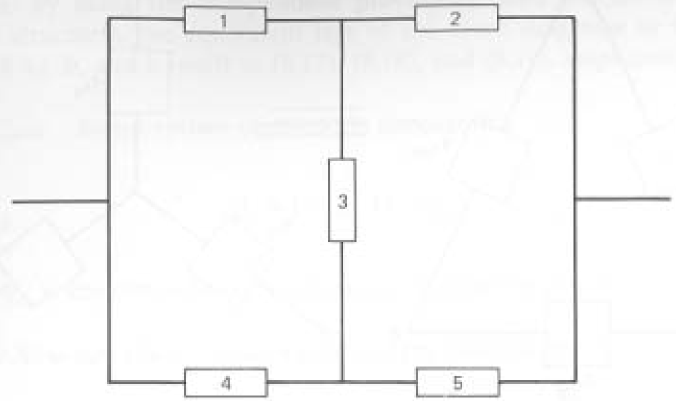


**Figure 8.2** A bridge network of dissimilar components.

for

$$K=\begin{cases}1, & \Phi_i=q_{oi}\\ 2, & \Phi_i=q_{si}\end{cases}$$

As shown in Figure 8.2, the bridge network is composed of five elements, $i=1,2,\ldots,5$, where the element number 3 is known as the critical element.

## 8.4 DELTA-STAR TRANSFORMATION TECHNIQUE

The reliability evaluation of series, parallel, and series-parallel networks is widely discussed. To evaluate the reliability of a bridge, or other such complex structures, the theories in the literature are difficult to apply. The delta-star transformation [8] is a simple approach for such problems. This technique transforms a complex structure to a series and parallel form. Thereon the network reduction technique may be applied to obtain reliability of transformed configuration. The technique introduces a small error, which can be neglected for practical purposes.

Transformations are carried out in terms of both of the failure modes instead of simply reliability or unreliability as is the case for a two-state device structure.

The resulting delta-star transformation formulas are developed by finding the leg equivalent, as illustrated by Figure 8.3.

### 8.4.1 *Open-Failure Mode*

The delta-star leg equivalents are obtained in the same manner as the simpler two-state component case. Figure 8.4 illustrates the leg equivalents for the open-mode failure case.

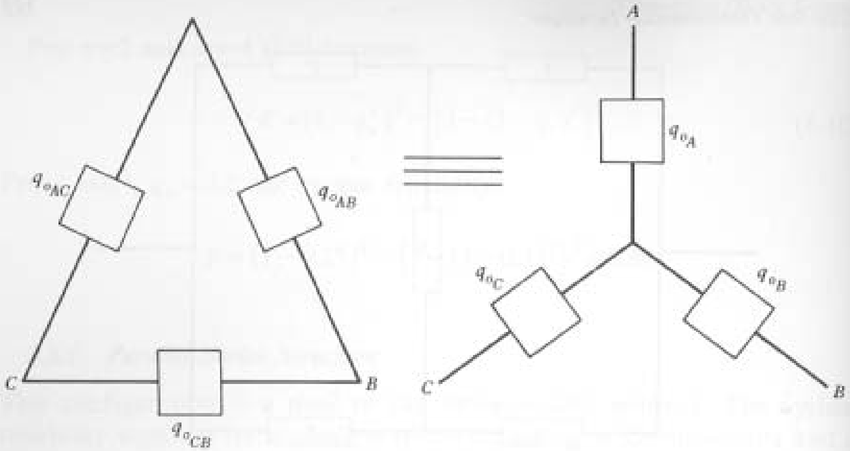


**Figure 8.3** A delta-star equivalent for the open-mode failure.

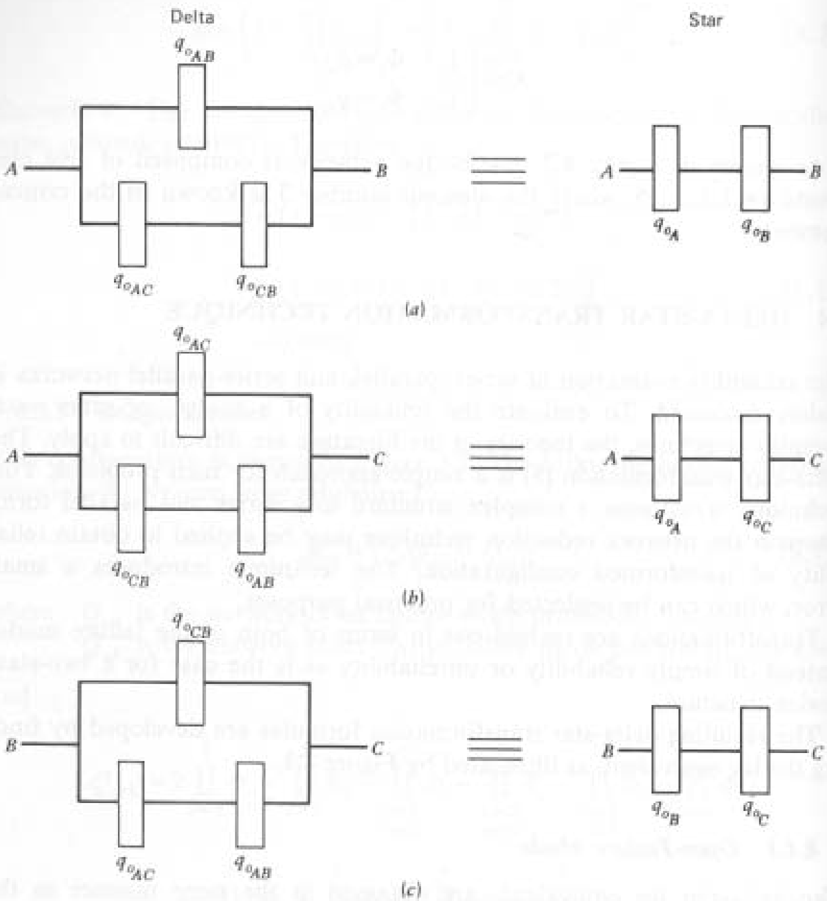


**Figure 8.4** Delta-star equivalent legs.





Again, by using the independent probability laws for the series and parallel structures, the equivalent legs of the block diagrams as shown in Figure 8.4*a*, *b*, and *c* result in (8.17), (8.18), and (8.19), respectively:

*Series Case.* Series system open-mode unreliability

$$Q_o = 1 - \prod_{i=1}^{n} (1 - q_{oi}) \tag{8.15}$$

Where $q_{oi}$ is the components' open-mode unreliability, $i = 1, n$.

*Parallel Structure Case.* Open-mode system unreliability

$$Q_o = \prod_{i=1}^{n} q_{oi} \tag{8.16}$$

With the aid of (8.15) and (8.16) the equivalent legs of the block diagrams are transformed, respectively, to the following:

$$1-(1-q_{o_A})(1-q_{o_C}) = \left[1-(1-q_{o_{CB}})(1-q_{o_{AB}})\right] q_{o_{AC}} \tag{8.17}$$

$$1-(1-q_{o_A})(1-q_{o_B}) = \left[1-(1-q_{o_{AC}})(1-q_{o_{CB}})\right] q_{o_{AB}} \tag{8.18}$$

$$1-(1-q_{o_B})(1-q_{o_C}) = \left[1-(1-q_{o_{AC}})(1-q_{o_{AB}})\right] q_{o_{CB}} \tag{8.19}$$

From these simultaneous equations result the following delta-star conversion equations:

$$q_{o_A} = 1 - \left[ \frac{\left[1-\{1-(1-q_{o_{CB}})(1-q_{o_{AB}})\}q_{o_{AC}}\right]\left[1-\{1-(1-q_{o_{AC}})(1-q_{o_{CB}})\}q_{o_{AB}}\right]}{\left[1-\{1-(1-q_{o_{AC}})(1-q_{o_{AB}})\}q_{o_{CB}}\right]} \right]^{1/2} \tag{8.20}$$

$$q_{o_B} = 1 - \left[ \frac{\left[1-\{1-(1-q_{o_{AC}})(1-q_{o_{CB}})\}\right]\left[1-\{1-(1-q_{o_{AC}})(1-q_{o_{AB}})\}q_{o_{CB}}\right]}{\left[1-\{1-(1-q_{o_{CB}})(1-q_{o_{AB}})\}q_{o_{AC}}\right]} \right]^{1/2} \tag{8.21}$$

$$q_{o_C} = 1 - \left[ \frac{\left[1-\{1-(1-q_{o_{CB}})(1-q_{o_{AB}})\}q_{o_{AC}}\right]\left[1-\{1-(1-q_{o_{AC}})(1-q_{o_AB})\}q_{o_{CB}}\right]}{\left[1-\{1-(1-q_{o_{AC}})(1-q_{o_{CB}})\}q_{o_{AB}}\right]} \right]^{1/2} \tag{8.22}$$

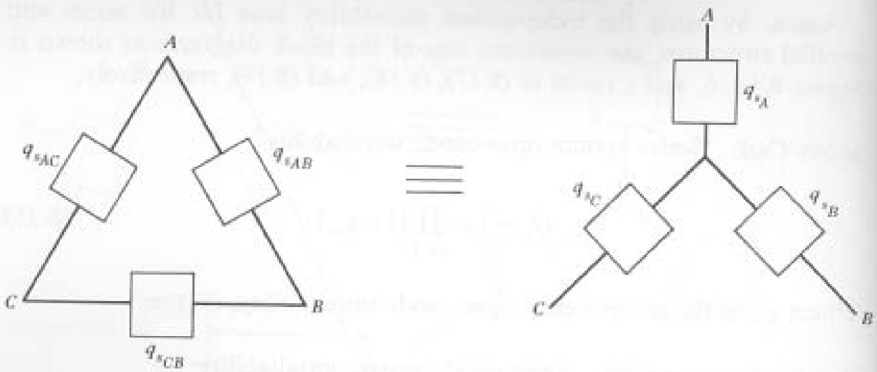


**Figure 8.5** A short-failure delta-star transformation.

### 8.4.2 *Short-Failure Mode*

Similarly, as for the open-failure mode, Figures 8.5 and 8.6 show the short-failure mode equivalent configurations.

Again, with aid of the independent probability laws for parallel and series structures, (8.25)–(8.27) are obtained from their equivalent corresponding legs of the block diagrams of Figure 8.6*a–c*.

*Series Case.* System short-mode unreliability

$$Q_s = \prod_{i=1}^{n} q_{si} \tag{8.23}$$

where $q_{si}$ is the components' short-mode unreliability, $i = 1, n$.

*Parallel Structure Case.*

$$Q_s = 1 - \prod_{i=1}^{n} (1 - q_{si}) \tag{8.24}$$

With applications of (8.23) and (8.24) to the equivalent legs of the block diagrams of Figure 8.6*a–c* the corresponding equations become

$$q_{s_A} q_{s_C} = 1 - (1 - q_{s_{CB}} q_{s_{AB}})(1 - q_{s_{AC}}) \tag{8.25}$$

$$q_{s_A} q_{s_B} = 1 - (1 - q_{s_{AC}} q_{s_{CB}})(1 - q_{s_{AB}}) \tag{8.26}$$

$$q_{s_B} q_{s_C} = 1 - (1 - q_{s_{AC}} q_{s_{AB}})(1 - q_{s_{CB}}) \tag{8.27}$$

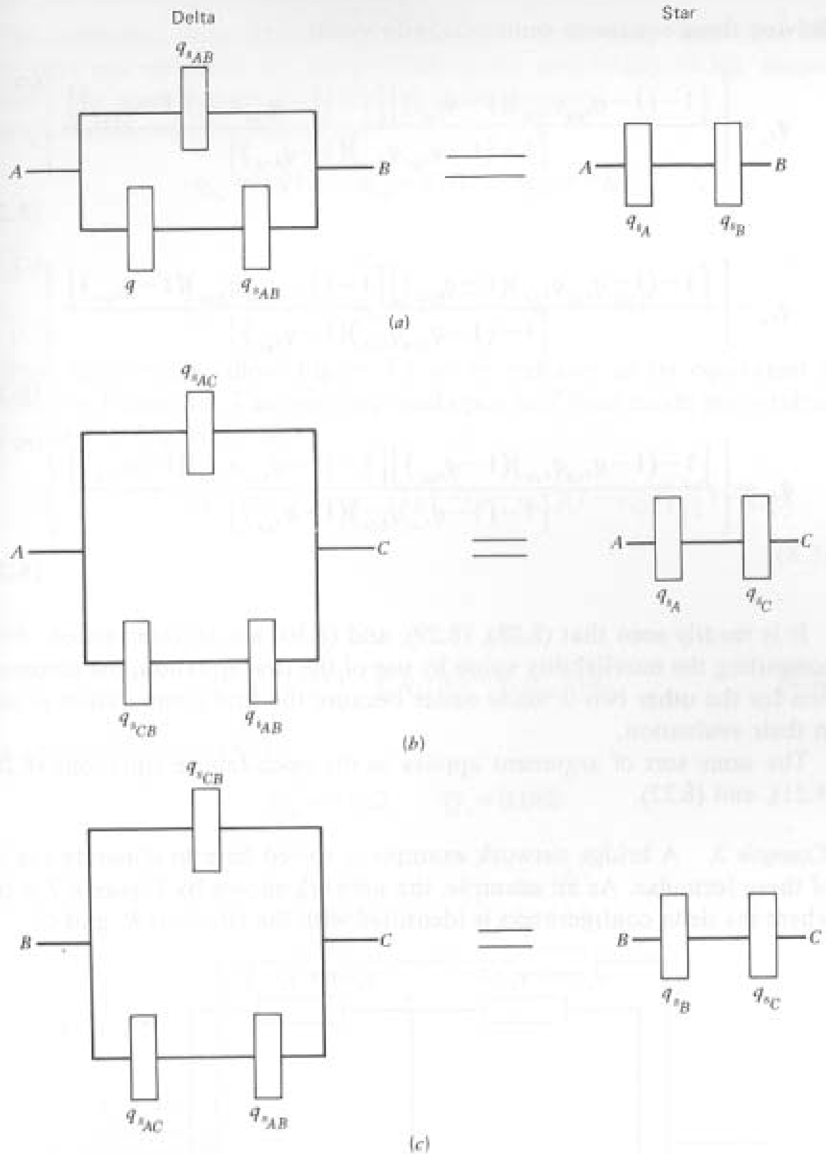


**Figure 8.6** Delta-star equivalent legs.

Solving these equations simultaneously yields

$$q_{s_A} = \left[ \frac{\left[1-(1-q_{s_{CB}}q_{s_{AB}})(1-q_{s_{AC}})\right]\left[1-(1-q_{s_{AC}}q_{s_{CB}})(1-q_{s_{AB}})\right]}{\left[1-(1-q_{s_{AC}}q_{s_{AB}})(1-q_{s_{CB}})\right]} \right]^{1/2} \tag{8.28}$$

$$q_{s_B} = \left[ \frac{\left[1-(1-q_{s_{AC}}q_{s_{AB}})(1-q_{s_{CB}})\right]\left[1-(1-q_{s_{AC}}q_{s_{CB}})(1-q_{s_{AB}})\right]}{\left[1-(1-q_{s_{CB}}q_{s_{AB}})(1-q_{s_{AC}})\right]} \right]^{1/2} \tag{8.29}$$

$$q_{s_C} = \left[ \frac{\left[1-(1-q_{s_{CB}}q_{s_{AB}})(1-q_{s_{AC}})\right]\left[1-(1-q_{s_{AC}}q_{s_{AB}})(1-q_{s_{CB}})\right]}{\left[1-(1-q_{s_{AC}}q_{s_{CB}})(1-q_{s_{AB}})\right]} \right]^{1/2} \tag{8.30}$$

It is readily seen that (8.28), (8.29), and (8.30) are all interrelated. After computing the unreliability value by use of the first equation, the computation for the other two is made easier because the first computation is used in their evaluation.

The same sort of argument applies to the open-failure equations (8.20), (8.21), and (8.22).

*Example 5.* A bridge network example is solved here to illustrate the use of these formulas. As an example, the network shown by Figure 8.7 is one where the delta configuration is identified with the labels *A*, *B*, and *C*.

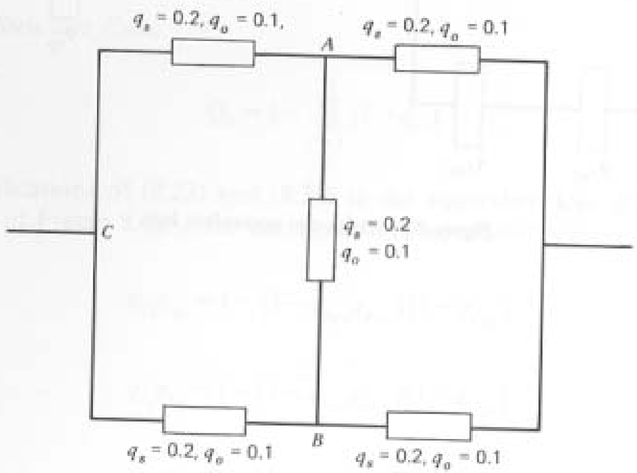


**Figure 8.7** A three-state bridge structure.



Its equivalent open and short failure mode probability values for this situation are obtained by using (8.20)–(8.22) and (8.28)–(8.30), respectively. The numerical results obtained are as follows:

Open-mode failure probability:

$$q_{o_A}=0.01 \qquad q_{o_B}=0.01 \qquad q_{o_C}=0.01$$

and

$$q_{s_A}=0.482 \qquad q_{s_B}=0.482 \qquad q_{s_C}=0.482$$

These relationships allow Figure 8.7 to be redrawn as its equivalent as shown by Figure 8.8. The resulting total open and short mode probabilities of failure for Figure 8.8 are

$$Q_o = 1-\left[1-\{(1-q_{s_1})(1-q_{o_A})\}\{1-(1-q_{o_B})(1-q_{o_2})\}\right]\left[1-q_{o_C}\right] \tag{8.31}$$

and

$$Q_s = \left[1-(1-q_{s_1}q_{s_A})(1-q_{s_2}q_{s_B})\right]q_{s_C} \tag{8.32}$$

By using (8.31) and (8.32)

$$Q_o=0.022 \qquad Q_s=0.088$$

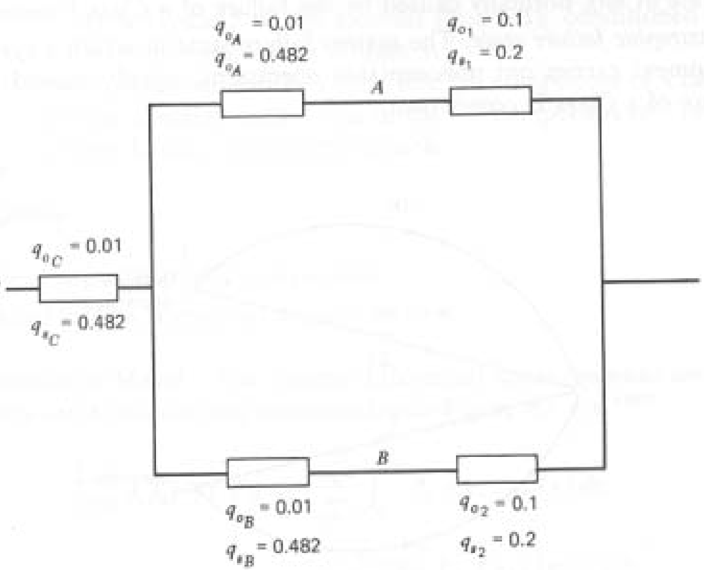


**Figure 8.8** A transformed bridge structure.

thereby giving bridge reliability

$$R_T = 1 - Q_o - Q_s = 0.89$$

## 8.5 REPAIRABLE THREE-STATE DEVICE SYSTEMS

This section presents several mathematical models of repairable systems. Most of these models are available in the referenced literature.

### 8.5.1 *Analysis of a Three-State System with Two Types of Components*

This model is developed by using the supplementary variables technique [39–41]. The three-state model [9] discussed in this section is shown in Figure 8.9. The components of this system are divided into classes (i.e., Class I and II). If any one component of Class I fails, the system will experience a complete system failure. A component failure of Class II will cause a catastrophic system failure. Some typical examples of such a system are automatic machines, fluid flow valves, a rotational mechanical system that jams so that rotation is blocked, a shaft that shears so that an input rotation causes no output rotation, and an electrical or electronic system.

System states are defined as follows:

1. *Normal state*. The successful functioning of a device.
2. *Complete failure state*. Total system failure (i.e., the machine does not operate at all), normally caused by the failure of a Class I component.
3. *Catastrophic failure state*. The system failure state in which a system or equipment carries out unacceptable operations, usually caused by the failure of a Class II component.

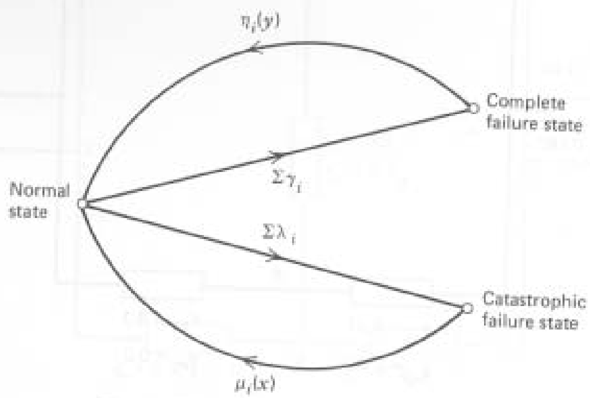


**Figure 8.9** A three-state Markov model.



Suppose an automatic machine carries out some operations on assembly line items. The automatic machine is composed of many component parts; therefore, the components of the machine can be divided into two classes (i.e., Class I and II). A component failure of Class I causes the complete failure or breakdown of the automatic machine. A failure of any one component of Class II will cause a catastrophic failure of the automatic system (this type of failure will initiate some unwanted operations on the assembly line items).

Obviously, to restore the automatic machine back to its normal state, repair is necessary. Repair times are arbitrarily distributed.

The following notations and definitions are used to formulate this Markov Model:

$P_o(t)$ = the probability of the system being in its normal mode at time $t$.

$P_{1,i}(y,t)\Delta$ = the probability that at time $t$, the system which has failed, because of the failure of its $i$th component in Class I, is being repaired and the elapsed repair time lies in the interval $(y, y+\Delta t)$ for $i = 1,2,3,n$.

$P_{2,i}(x,t)\Delta$ = the probability that at time $t$ the system that has failed, because of the failure of its $i$th component in Class II, is being repaired and the elapsed repair time lies in the interval $(x, x+\Delta t)$ for $1 = 1,2,3,n$.

$\eta_i(y)\Delta \equiv$ the first-order probability, that the $i$th component of Class I is repaired in the interval $(y, y+\Delta)$, conditioned that it was not repaired up to time $y$.

$\mu_i(x)\Delta \equiv$ the first-order probability, that the $i$th component of Class II is repaired in the interval $(x, x+\Delta)$, conditioned that it was not repaired up to time $x$.

$\lambda_i$ = the constant failure rate of the $i$th component of Class II.

$\gamma_i$ = the constant failure rate of the $i$th component of Class I.

$s$ = the Laplace transform variable.

**Assumptions**

1. Failures are statistically independent.
2. A failed system is restored as good as new.

*A Mathematical Model.* The integro-differential equations (and associated boundary-initial conditions) associated with Figure 8.9 are

$$\left[\frac{\partial}{\partial t} + \lambda + \gamma\right] \cdot P_0(t) = \sum_{i=1}^{n} \int_0^\infty P_{2,i}(x,t)\mu_i(s)\,dx + \sum_{i=1}^{n} \int_0^\infty P_{1,i}(y,t)\eta_i(y)\,dy \tag{8.33}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\eta_i(y)\right]\cdot P_{1,i}(y,t)=0 \quad (8.34)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_i(x)\right]\cdot P_{2,i}(x,t)=0 \quad (8.35)$$

$$P_{1,i}(0,t)=\gamma_i\cdot P_0(t) \qquad P_{2,i}(0,t)=\lambda_i\cdot P_0(t)$$

$P_0(0)=1$, at $t=0$ other initial condition probabilities are zero, where

$$\lambda=\sum_{i=1}^{n}\lambda_i \qquad \gamma=\sum_{i=1}^{n}\gamma_i$$

Solving the above integro-differential equations by Laplace transforms and integration (including some substitutions) will yield

$$P_0(s)=\frac{1}{\left[(s+\lambda+\gamma)-\sum_{i=1}^{n}\lambda_i G_{2,i}(s)-\sum_{i=1}^{n}\gamma_i G_{1,i}(s)\right]} \quad (8.36)$$

where

$$G_{1,i}(s)=\int_0^{\infty}e^{-sy}\eta_i(y)\exp\left(-\int_0^{y}\eta_i(y)\,dy\right)dy$$

$$G_{2,i}(s)=\int_0^{\infty}e^{-sx}\mu_i(s)\exp\left(-\int_0^{x}\mu_i(x)\,dx\right)dx$$

Since

$$P_{1,i}(s)=\int_0^{\infty}P_{1,i}(y,s)\,dy \quad (8.37)$$

$$P_{2,i}(s)=\int_0^{\infty}P_{2,i}(x,s)\,dx, \quad (8.38)$$

where $P_{1,i}(s)$, $P_{2,i}(s)$ are the Laplace transform of probabilities $P_{1,i}(t)$, $P_{2,i}(t)$ that the system is under repair due to the failure of the $i$th component in Classes I and II, respectively. Therefore,

$$P_{j,i}(s)=P_0(s)\left\{\frac{1-G_{j,i}(s)}{s}\right\}k_j \quad (8.39)$$

$$\text{for } j=1,2;\ i=1,2,3,n;\ k_1=\gamma_i;\ k_2=\lambda_i$$

The Laplace transforms of probabilities $P_1(t)$ and $P_2(t)$, that system is under repair due to the failure of any one of Classes I and II components, respectively, are



$$P_j(s)=\sum_{i=1}^{n}P_{j,i}(s) \qquad \text{for } j=1,2 \quad (8.40)$$

Substituting (8.39) into (8.40) yields

$$P_j(s)=\sum_{i=1}^{n}P_0(s)\left\{\frac{1-G_{j,i}(s)}{s}\right\}kj$$

$$\text{for } j=1,2;\ k_1=\gamma_i;\ k_2=\lambda_i \quad (8.41)$$

Therefore, for given repair probability density functions $G_{j,i}(t)$, the state probabilities $P_0(t)$, $P_j(t)$ can be obtained by simply taking the inverse Laplace transform of (8.36) and (8.41), respectively.

The steady-state solution, if it exists, of (8.36) and (8.41) can be obtained by employing Abel's Theorem to Laplace transform,

$$\lim_{s\to 0}sf(s)=\lim_{t\to\infty}f(t). \quad (8.42)$$

Mean time to system failure (MTSF) (if exists) can be obtained from

$$\text{MTSF}=\lim_{s\to 0}P_0(s) \quad (8.43)$$

More detailed analysis of similar models using the method of supplementary variables are presented in references 39–41.

### *8.5.2 A Repairable Three-State Device with Constant Failure and Repair Rates*

This model [30] is a special case of the model presented in Section 8.5.1. The system transition diagram is shown in Figure 8.10.

**Assumptions**

1. Failures are statistically independent.
2. The repaired system is as good as new.
3. Repair and failure rates are constant.

**Notation**

$P_i(t)$ = the probability of the state in question, at time $t$; $i=0,1,2$
$\lambda$ = the constant failure rate in question
$\mu$ = the constant repair rate in question

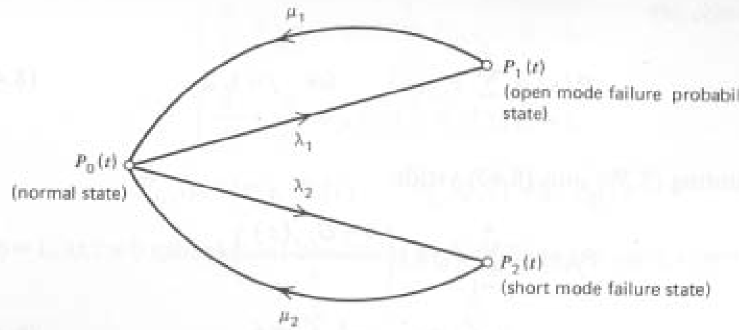


**Figure 8.10** A repairable Markov model.

From Figure 8.10, the resulting differential equations are:

$$\frac{dP_0(t)}{dt} = -(\lambda_1+\lambda_2)P_0(t) = \mu_1 P_1(t) + \mu_2 P_2(t) \tag{8.44}$$

$$\frac{dP_1(t)}{dt} = P_0(t)\lambda_1 - \mu_1 P_1(t) \tag{8.45}$$

$$\frac{dP_2(t)}{dt} = \lambda_2 P_0(t) - \mu_2 P_2(t) \tag{8.46}$$

$$P_0(0)=1 \quad P_1(0)=P_2(0)=0$$

The Laplace transform of (8.44)–(8.46) yields

$$(s+\lambda_1+\lambda_2)P_0(s) - \mu_1 P_1(s) - \mu_2 P_2(s) = 1 \tag{8.47}$$

$$-\lambda_1 P_0(s) + (s+\mu_1)P_1(s) + 0P_2(s) = 0 \tag{8.48}$$

$$-\lambda_2 P_0(s) + 0P_1(s) + (s+\mu_2)P_2(s) = 0 \tag{8.49}$$

The coefficient of the above simultaneous equations can be written as follows:

$$\left(\begin{array}{ccc|c} (s+\lambda_1+\lambda_s) & -\mu_1 & -\mu_2 & 1 \\ -\lambda_1 & (s+\mu_1) & 0 & 0 \\ -\lambda_2 & 0 & (s+\mu_2) & 0 \end{array}\right)$$



∴The solution by Cramer's rule yields:

$$P_0(s) = \frac{(s+\mu_1)(s+\mu_2)}{s[s^2+s(\mu_1+\mu_2+\lambda_1+\lambda_2)+(\mu_1\mu_2+\lambda_1\mu_2+\lambda_1\mu_2+\lambda_2\mu_1)]} \tag{8.50}$$

$$P_1(s) = \frac{\lambda_1(s+\mu_1)}{s[s^2+s(\mu_1+\mu_2+\lambda_1+\lambda_2)+\mu_1\mu_2+\lambda_1\mu_2+\lambda_2\mu_1]} \tag{8.51}$$

$$P_2(s) = \frac{\lambda_2(s+\mu_1)}{s[s^2+s(\mu_1+\mu_2+\lambda_1+\lambda_2)+(\mu_1\mu_2+\lambda_1\mu_2+\lambda_2\mu_1)]} \tag{8.52}$$

The roots of the denominators of (8.50)–(8.52) become

$$k_1, k_2 = \frac{-(\mu_1+\mu_2+\lambda_1+\lambda_2) \pm \sqrt{(\mu_1+\mu_2+\lambda_1+\lambda_2)^2 - 4(\mu_1\mu_2+\lambda_1\mu_2+\lambda_2\mu_2)}}{2}$$

Now, (8.50)–(8.52) can be expanded in a partial fraction form

$$P_0(s) = \frac{(s+\mu_1)(s+\mu_2)}{s(s-k_1)(s-k_2)}$$

$$= \frac{\mu_1\mu_2}{k_1k_2}\frac{1}{s} + \frac{(k_1+\mu_1)(k_1+\mu_2)}{k_1(k_1-k_2)}\frac{1}{s-k_1} - \frac{(k_2+\mu_1)(k_2+\mu_2)}{k_2(k_1-k_2)}\frac{1}{(s-k_2)} \tag{8.53}$$

$$P_1(s) = \frac{\lambda_1(s+\mu_2)}{s(s-k_1)(s-k_2)}$$

$$= \frac{\lambda_1\mu_2}{k_1k_2} + \frac{(\lambda_1k_1+\lambda_1\mu_2)}{k_1(k_1-k_2)}\frac{1}{(s-k_1)} - \frac{(\mu_2+k_2)\lambda_1}{k_2(k_1-k_2)}\frac{1}{(s-k_2)} \tag{8.54}$$

$$P_2(s) = \frac{\lambda_2(s+\mu_1)}{s(s-k_1)(s-k_2)}$$

$$= \frac{\lambda_2\mu_1}{k_1k_2} + \frac{(\lambda_2k_1+\lambda_2\mu_1)}{k_1(k_1-k_2)}\frac{1}{(s-k_1)} - \frac{(\mu_1+k_2)\lambda_2}{k_2(k_1-k_2)}\frac{1}{(s-k_2)} \tag{8.55}$$

In time domain, (8.53) and (8.55) become

$$P_0(t)=\frac{\mu_1\mu_2}{k_1k_2}+\left\{\frac{(k_1+\mu_1)(k_1+\mu_2)}{k_1(k_1-k_2)}\right\}e^{k_1t}-\left\{\frac{(k_2+\mu_1)(k_2+\mu_2)}{k_2(k_1-k_2)}\right\}e^{k_2t} \tag{8.56}$$

$$P_1(t)=\frac{\lambda_1\mu_2}{k_1k_2}+\left\{\frac{\lambda_1k_1+\lambda_1\mu_2}{k_1(k_1-k_2)}\right\}e^{k_1t}-\left\{\frac{(\mu_2+k_2)\lambda_1}{k_2(k_1-k_2)}\right\}e^{k_2t} \tag{8.57}$$

$$P_2(t)=\frac{\lambda_2\mu_1}{k_1k_2}+\left\{\frac{\lambda_2k_1+\lambda_2\mu_1}{k_1(k_1-k_2)}\right\}e^{k_1t}-\frac{(\mu_1+k_2)\lambda_2}{k_2(k_1-k_2)}e^{k_2t} \tag{8.58}$$

Since

$$k_1k_2=\mu_1\mu_2+\lambda_1\mu_2+\lambda_2\mu_1$$

$$k_1+k_2=-(\mu_1+\mu_2+\lambda_1+\lambda_2)$$

therefore, the addition of (8.56)–(8.58) will yield unity, that is,

$$P_0(t)+P_1(t)+P_2(t)=1$$

The equipment availability is

$$\text{Availability}=P_0(t)=\frac{\mu_1\mu_2}{k_1k_2}+\left\{\frac{(k_1+\mu_1)(k_1+\mu_2)}{k_1(k_1-k_2)}\right\}e^{k_1t}-\left\{\frac{(k_2+\mu_1)(k_2+\mu_2)}{k_2(k_1-k_2)}\right\}e^{k_2t}$$

The availability expression is valid if and only if $k_1$ and $k_2$ are negative. As $t$ becomes very large, the steady-state availability equation can be expressed as

$$\lim_{t\to\infty}P_0(t)=\frac{\mu_1\mu_2}{k_1k_2} \tag{8.59}$$

### 8.5.3 *A Mixed Markov Model with Two Three-State Devices (Master-Slave Relationship)*

This mixed Markov model [34] has the two units modeled in series. One device has normal, partial, and catastrophic states and the other has normal, open, and closed mode states (Type II). Repairs are performed only when an equipment fails in its partial mode.



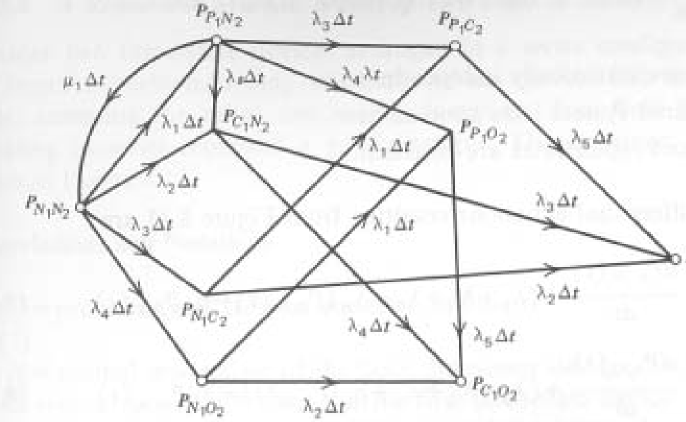


**Figure 8.11** A master-slave Markov model.

A typical example of such a system is a fluid flow valve commanded from an instrumentation control panel where the control panel represents the first type of device (Master) and the fluid flow valve represents the second type (Slave). Such practical examples are numerous and may often be encountered in a modern electrical power station. The transition diagram for this case is shown in Figure 8.11.

#### Abbreviations and Notations

$P(t)$ = probability of the state in question, at time $t$

$(\cdot)(\cdot)$

- $N_i$ = normal mode state of the three-state devices (i.e., master and slave), respectively, $i=1,2$.
- $C_1$ = catastrophic failure state of the "master" three-state device
- $C_2$ = closed mode failure state of the "slave" three-state device
- $P_1$ = partial failure state of the "master" three-state device
- $O_2$ = open mode failure state of the "slave" three-state device
- $\lambda_1$ = constant partial failure rate of the "master" three-state device
- $\lambda_2$ = constant catastrophic failure rate of the "master" three-state device
- $\lambda_5$ = constant failure rate from partial to catastrophic failure state of the "master" three-state device
- $\lambda_3$ = constant close mode failure rate of the slave three-state device
- $\lambda_4$ = constant open mode failure rate of the slave three-state device
- $\mu_1$ = constant repair rate of the master device
- $t$ = time
- $\Delta t$ = time interval

**Assumptions**

1. Failures are statistically independent.
2. The repaired system is as good as new.
3. Failure and repair rates are constant.

The state differential equations resulting from Figure 8.11 are

$$\frac{dP_{N_1N_2}(t)}{dt}+(\lambda_1+\lambda_2+\lambda_3+\lambda_4)P_{N_1N_2}(t)=\mu_1 P_{P_1N_2}(t) \qquad (8.60)$$

$$\frac{dP_{P_1N_2}(t)}{dt}+(\lambda_3+\lambda_4+\lambda_5+\mu_1)P_{P_1N_2}(t)=\lambda_1 P_{N_1N_2}(t) \qquad (8.61)$$

$$P_{N_1N_2}(0)=1 \qquad P_{P_1N_2}(0)=0$$

Solving the above differential Equations by Laplace transform yields

$$P_{P_1N_2}(t)=\left\{\frac{\lambda_1}{k_2-k_1}\right\}e^{k_2t}-\left\{\frac{\lambda_1}{k_2-k_1}\right\}e^{k_1t} \qquad (8.62)$$

$$P_{N_1N_2}(t)=\left\{1-\frac{(\lambda_3+\lambda_4+\lambda_5+\mu_1+k_2)}{(k_2-k_1)}\right\}e^{k_1t}$$

$$+\frac{\lambda_3+\lambda_4+\lambda_5+\mu_1+k_2}{k_2-k_1}e^{k_2t} \qquad (8.63)$$

where

$$k_1,k_2=\frac{-N\pm\sqrt{N^2-4AM}}{2A}$$

and

$$A=1$$

$$N=\lambda_1+\lambda_2+2\lambda_3+2\lambda_4+\lambda_5+\mu_1$$

$$M=\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_3^2+2\lambda_3\lambda_4+\lambda_1\lambda_4+\lambda_2\lambda_4+\lambda_4^2+\lambda_1\lambda_5+\lambda_2\lambda_5$$
$$+\lambda_3\lambda_5+\lambda_4\lambda_5+\lambda_2\mu_1+\lambda_3\mu_1+\lambda_4\mu_1$$

Therefore,

$$\text{System reliability}=P_{P_1N_2}(t)+P_{N_1N_2}(t) \qquad (8.64)$$



### 8.5.4 A Repairable Markov Model of Two Units in Series I

Consider two three-state devices arranged in a series configuration [34]. The repair is performed only when one of the devices fails in its closed mode, assuming the other one is still operating. Two fluid flow valves operating in series represent a good example. The transition diagram is shown in Figure 8.12.

**Abbreviations and Notations**

$P(t)$ = probability of state in question, at time $t$

$(\cdot)(\cdot)$

$N_i$ = normal mode state of the both three-state devices, $i=1,2$

$C_1$ = close mode failure state of the first three-state device

$C_2$ = close mode failure state of the second three-state device

$\mu_1$ = constant repair rate of the first three-state device

$\mu_2$ = constant repair rate of the second three-state device

$\lambda_1$ = constant close mode failure rate of the first three-state device

$\lambda_2$ = constant close mode failure rate of the second three-state = device

$\lambda_3$ = constant open mode failure rate of the first three-state device

$\lambda_4$ = constant open mode failure rate of the second three-state device

$t$ = time

$\Delta t$ = time interval

$s$ = Laplace Transform variable

**Assumptions**

1. Failures are statistically independent.
2. Repaired device is good as new.
3. Failure and repair rates are constant.

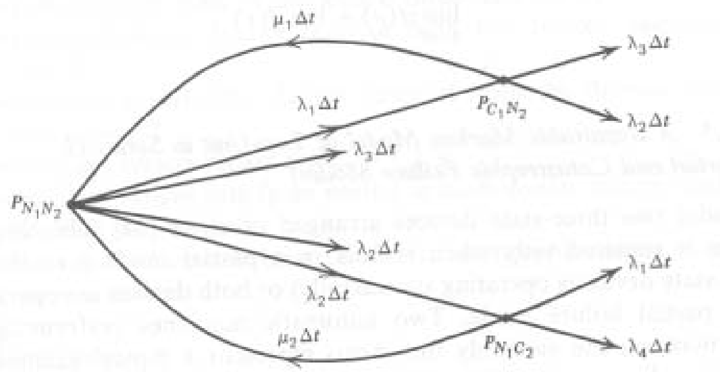


**Figure 8.12** A series two-unit repairable Markov model.

The differential Equations associated with Figure 8.12 are

$$\frac{dP_{N_1N_2}(t)}{dt}+(\lambda_1+\lambda_2+\lambda_3+\lambda_4)P_{N_1N_2}(t)=P_{C_1N_2}(t)\mu_1+P_{N_1C_2}(t)\mu_2 \tag{8.65}$$

$$\frac{dP_{C_1N_2}(t)}{dt}+(\lambda_2+\lambda_3+\mu_1)P_{C_1N_2}(t)=P_{N_1N_2}(t)\lambda_1 \tag{8.66}$$

$$\frac{dP_{N_1C_2}(t)}{dt}+(\lambda_1+\lambda_4+\mu_2)P_{N_1C_2}(t)=P_{N_1N_2}(t)\lambda_2 \tag{8.67}$$

$$P_{N_1N_2}(0)=1 \qquad P_{C_1N_2}(0)=0 \qquad P_{N_1C_2}(0)=0$$

The values of $P_{N_1N_2}(s)$, $P_{N_1C_2}(s)$, $P_{C_1N_2}(s)$ are obtained from the above differential equations:

$$P_{N_1N_2}(s)=\frac{(s+\lambda_2+\lambda_3+\mu_1)(s+\lambda_1+\lambda_4+\mu_2)}{\Delta} \tag{8.68}$$

where $\Delta=\begin{vmatrix}(s+\lambda_1+\lambda_2+\lambda_3+\lambda_4) & -\mu_1 & -\mu_2\\ -\lambda_1 & (s+\lambda_2+\lambda_3+\mu_1) & 0\\ -\lambda_2 & 0 & (s+\lambda_1+\lambda_4+\lambda_2)\end{vmatrix}$

$$P_{C_1N_2}(s)=\frac{\lambda_1(s+\lambda_1+\lambda_4+\mu_2)}{\Delta} \tag{8.69}$$

$$P_{N_1C_2}(s)=\frac{-\lambda_2(s+\lambda_2+\lambda_3+\mu_1)}{\Delta} \tag{8.70}$$

The steady-state solutions (if they exist) of (8.68)–(8.70) can be obtained by employing Abel's Theorem to Laplace Transform, that is,

$$\lim_{s\to 0} sf(s)=\lim_{t\to\infty} f(t) \tag{8.71}$$

### *8.5.5 A Repairable Markov Model of Two-Unit in Series II (Partial and Catastrophic Failure Modes)*

Consider two three-state devices arranged in series [34]. The three-state device is repaired only when it fails in a partial mode (i.e., the other three-state device is operating successfully) or both devices are operating in their partial failure mode. Two automatic machines performing some operations on the assembly line items represent a typical example. The transition diagram for this series configuration is shown in Figure 8.13.



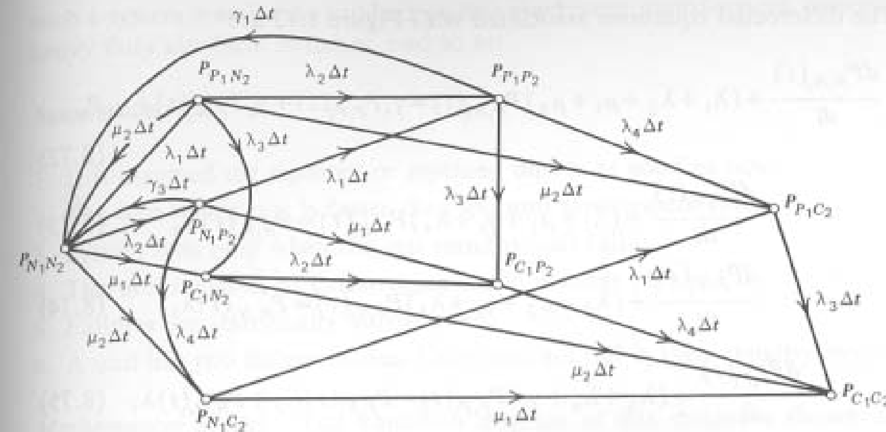

**Figure 8.13** A series two-unit repairable Markov model.

**Assumptions**

1. Failures are statistically independent.
2. Repaired system is as good as new.
3. Failure and repair rates are constant.

**Abbreviations and Notations**

$P(t)$ = probability of state in question at time $t$.

$(\cdot)(\cdot)$

- $N_i$ = normal state of both three-state devices, $i=1,2$
- $C_1$ = catastrophic failure state of the first three-state device
- $C_2$ = catastrophic failure state of the second three-state device
- $P_1$ = partial failure state of the first three-state device
- $P_2$ = partial failure state of the second three-state device
- $\lambda_i$ = constant partial failure rates of both the devices, respectively, $i=1,2$
- $\mu_i$ = constant catastrophic failure rates of both the devices, respectively, $i=1,2$
- $\gamma_i$ = constant system repair rates, $i=1,2,3$
- $\lambda_3$ = constant failure rate from partial to catastrophic failure mode of the first unit or device
- $\lambda_4$ = constant failure rate from partial to catastrophic failure mode of the second unit
- $t$ = time
- $\Delta t$ = time interval
- $s$ = Laplace Transform variable

The differential equations associated with Figure 8.13 are

$$\frac{dP_{N_1N_2}(t)}{dt}+(\lambda_1+\lambda_2+\mu_1+\mu_2)P_{N_1N_2}(t)=\gamma_1P_{P_1P_2}(t)+\gamma_2P_{P_1N_2}(t)+\gamma_3P_{N_1P_2} \tag{8.72}$$

$$\frac{dP_{N_1P_2}(t)}{dt}+(\lambda_1+\mu_1+\gamma_3+\lambda_4)P_{N_1P_2}(t)=P_{N_1N_2}(t)\lambda_2 \tag{8.73}$$

$$\frac{dP_{P_1N_2}(t)}{dt}+(\lambda_2+\mu_2+\gamma_2+\lambda_3)P_{P_1N_2}(t)=P_{N_1N_2}(t)\lambda_1 \tag{8.74}$$

$$\frac{dP_{P_1P_2}(t)}{dt}+(\lambda_3+\lambda_4+\gamma_1)P_{P_1P_2}(t)=P_{P_1N_2}(t)\lambda_2+P_{N_1P_2}(t)\lambda_1 \tag{8.75}$$

$P_{N_1N_2}(0)=1$, at $t=0$ other initial condition probabilities are zero.

The values of $P_{N_1N_2}(s)$, $P_{N_1P_2}(s)$, $P_{P_1N_2}(s)$, $P_{P_1P_2}(s)$ are obtained from the above differential equations:

$$P_{N_1N_2}(s)=\frac{-(\mu_1+\lambda_1+\gamma_3+\lambda_4)(s+\mu_2+\lambda_2+\gamma_2+\lambda_3)(s+\lambda_3+\lambda_4+\gamma_1)}{\Delta} \tag{8.76}$$

$$\Delta=\begin{vmatrix}(s+\lambda_1+\lambda_3+\mu_1+\mu_2) & -\gamma_1 & -\gamma_2 & \gamma_3\\ -\lambda_2 & 0 & 0 & (s+\mu_1+\lambda_1+\gamma_3+\lambda_4)\\ -\lambda_1 & 0 & (s+\mu_2+\lambda_2+\gamma_2+\lambda_3) & 0\\ 0 & (s+\lambda_3+\lambda_4+\gamma_1) & -\lambda_2 & -\lambda_1\end{vmatrix}$$

$$P_{P_1P_2}(s)=\frac{-\{\lambda_2\lambda_1(s+\mu_2+\lambda_2+\gamma_2+\lambda_3)\}+(\mu_1+\lambda_1+\gamma_3+\lambda_4)\lambda_1\lambda_2}{\Delta} \tag{8.77}$$

$$P_{P_1N_2}(s)=\frac{-(s+\mu_1+\lambda_1+\gamma_3+\lambda_4)(s+\lambda_3+\lambda_4+\gamma_1)\lambda_1}{\Delta} \tag{8.78}$$

$$P_{N_1P_2}(s)=\frac{-\lambda_2(s+\mu_2+\lambda_2+\gamma_2+\lambda_3)(s+\lambda_3+\lambda_4+\gamma_1)}{\Delta} \tag{8.79}$$

### 8.5.6 *A Two-Failure-Mode System with Cold Stand-By Units*

Mathematical model [16] presents a system with two failure mode units and $N$ stand-by units. The operational unit can be repaired at one of its failure modes. This may be regarded as a minor failure mode, in a case where the on-line failures can be repaired at the place of equipment installation or the unit repair time is less than the unit replacement time. When the unit repair is costly and time consuming, the failed unit is replaced with one of the stand-by units. Some of the typical examples of



such a system may be the production line machinery, transformers, motors, heavy duty electrical switches, and so on.

**Assumptions**

1. It is assumed the repaired or replaced unit is as good as new.
2. The unit repair rate is faster than the unit replacement rate.
3. System fails only when the last standby unit fails.
4. The unit failed in its catastrophic mode is never repaired.
5. Failures are statistically independent.
6. A unit has two failure modes. Units can not fail in their standby mode.

*Mathematical Model.* The transition diagram of this system is shown in Figure 8.14. The following definitions and notations are used to formulate this mathematical model:

$N$ = number of identical standby units
$n$ = last state number of the system
$\mu_i$ = constant replacement and repair rates, respectively, of the operational unit for $i=1,2$ and $\mu_2>\mu_1$
$\lambda_1$ = constant unit replacement mode failure rate
$\lambda_2$ = constant noncatastrophic mode (repairable mode) failure rate
$t$ = time
$s$ = Laplace transform variable
$P_0(t)$ = unit operational mode probability at time $t$
$P_1(t)$ = unit repairable mode probability at time $t$
$P_{k-i}(t)$ = unit failure, system operational and, system repairable mode probabilities at time $t$, for $i=2,1,0$ respectively, and $k=4,7,10,\ldots,(n-1)$
$P_n(t)$ = system failure mode probability at time $t$

The system differential equations for the Figure 8.14 model are

$$P_0'(t)=-(\lambda_1+\lambda_2)P_0(t)+P_1(t)\mu_2 \tag{8.80}$$

$$P_1'(t)=\mu_2P_1(t)+P_0(t)\lambda_1 \tag{8.81}$$

$$P_{k-2}'(t)=-\mu_1P_{k-2}(t)+P_{k-4}(t)\lambda_2 \tag{8.82}$$

$$P_{k-1}'(t)=-(\lambda_1+\lambda_2)P_{k-1}(t)+P_k(t)\mu_2+P_{k-2}(t)\mu_1 \tag{8.83}$$

$$P_k'(t)=-\mu_2P_k(t)+P_{k-1}(t)\lambda_1 \tag{8.84}$$

$$P_n'(t)=P_{k-1}(t)\lambda_2 \tag{8.85}$$

for $k=4,7,10,\ldots,(n-1)$

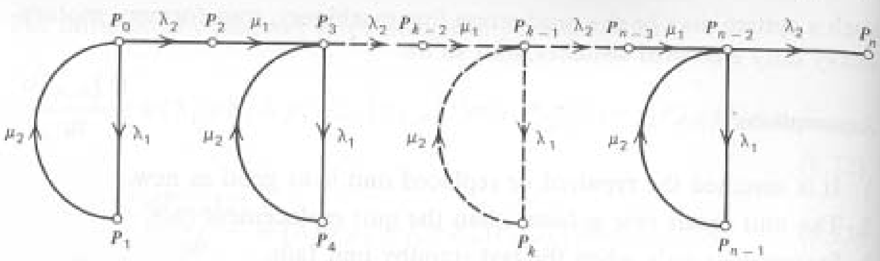


Figure 8.14 A transition diagram.

At $P_0(0)=1$, other initial condition probabilities are equal to zero.

$$n=3(N+1)-1 \quad \text{for} \quad N>1 \tag{8.86}$$

where the prime denotes differentiation with respect to time $t$. The Laplace transforms of the solution are

$$P_0(s)=\frac{\{s+\mu_2\}}{\{(s+\mu_2)(s+\lambda_1+\lambda_2)-\lambda_1\mu_2\}} \tag{8.87}$$

$$P_1(s)=\frac{\{P_0(s)\lambda_1\}}{\{s+\mu_2\}} \tag{8.88}$$

$$P_{k-2}(s)=\frac{P_{k-4}(s)\lambda_2}{s+\mu_1} \tag{8.89}$$

$$P_{k-1}(s)=\frac{P_k(s)\mu_2+P_{k-2}(s)\mu_1}{s+\lambda_1+\lambda_2} \tag{8.90}$$

$$P_k(s)=\frac{P_{k-1}(s)\lambda_1}{s+\mu_2} \tag{8.91}$$

$$P_n(s)=\frac{P_{n-1}(s)\lambda_2}{s}. \tag{8.92}$$

### 8.5.7 Availability Analysis of a Two-Failure Modes System with Nonrepairable Stand-by Units

This model considers a system containing $N$ identical units of which one is functioning and $(N-1)$ are standbys. As soon as the operational unit fails in any one of the two failure modes, it is replaced by one of the $(N-1)$ standby units. The system functions until the last standby unit is operational. The transition diagram of the Markov model is shown in Figure 8.15.



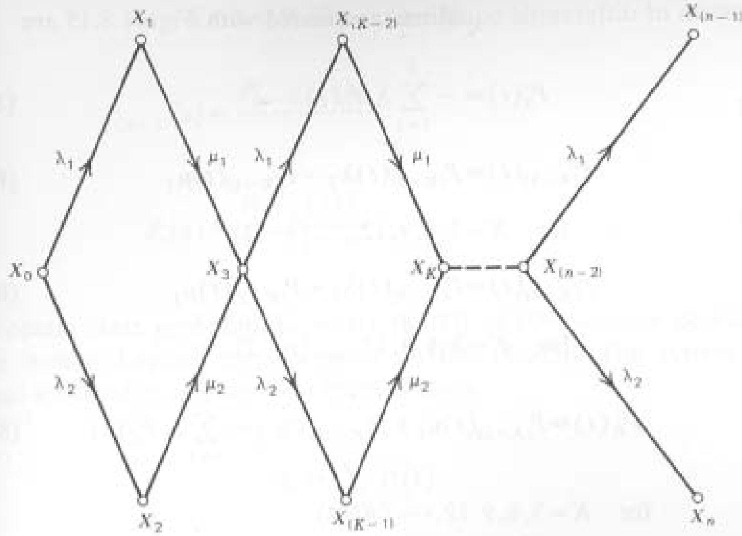


Figure 8.15 An $(N-1)$ units standby Markov model.

#### Notation

$X_i$ = system operational (i.e., for $i=0,3,6,9,\ldots(n-2)$), failure mode type I (i.e., for $i=1,4,7,10,\ldots,(n-1)$ and failure mode type II (i.e., for $i=2,5,8,\ldots,n$) states

$P_i(t)$ = system operational (i.e., for $i=0,3,6,9,\ldots,(n-2)$, failure mode type I (i.e., for $i=1,4,7,10,\ldots,(n-1)$, and failure mode type II (i.e., for $i=2,5,8,\ldots,n$) probabilities at time $t$

$\lambda_i$ = constant type I and type II failure mode failure rates of the operational unit, respectively (i.e., for $i=1,2$)

$\mu_i$ = constant type I and type II failure mode state replacement rates of the failed unit, respectively (i.e., for $i=1,2$)

$n$ = number of system states

$N$ = number of units in the system (i.e., the operational unit plus standby units)

$t$ = time

$s$ = Laplace transform variable

#### Assumptions

1. Failures are statistically independent.
2. Restored unit is as good as new.
3. Cold standby units cannot fail.
4. Failed unit is never repaired.

The system of differential equations associated with Figure 8.15 are

$$P_0'(t) = -\sum_{i=1}^{2} \lambda_i P_0(t) \tag{8.93}$$

$$P_{(K-1)}'(t) = P_{(K-3)}(t)\lambda_2 - P_{(K-1)}(t)\mu_2 \tag{8.94}$$

for $K = 3, 6, 9, 12, \ldots, (n-2)$

$$P_{(K-2)}'(t) = P_{(K-3)}(t)\lambda_1 - P_{(K-2)}(t)\mu_1 \tag{8.95}$$

for $K = 3, 6, 9, 12, \ldots, (n-2)$

$$P_K'(t) = P_{(K-2)}(t)\mu_1 + P_{(K-1)}(t)\mu_2 - \sum_{i=1}^{2} \lambda_i P_K(t) \tag{8.96}$$

for $K = 3, 6, 9, 12, \cdots (n-2)$

$$P_{(n-1)}'(t) = P_{(n-2)}(t)\lambda_1 \tag{8.97}$$

$$P_n'(t) = P_{(n-2)}(t)\lambda_2 \tag{8.98}$$

At $P_0(0) = 1$, other initial condition probabilities are zero.

$$n = (3N-1) \quad \text{for } N \geqslant 1 \tag{8.99}$$

where the prime denotes the derivative with respect to time $t$. Solutions to the above system of differential equations in the $s$ domain are

$$P_0(s) = \frac{1}{s + \sum_{i=1}^{2} \lambda_i} \tag{8.100}$$

$$P_{(K-1)}(s) = \frac{P_{(K-3)}(s)\lambda_2}{s + \mu_2} \tag{8.101}$$

for $K = 3, 6, 9, 12, \ldots, (n-2)$

$$P_{(K-2)}(s) = \frac{P_{(K-3)}(s)\lambda_1}{s + \mu_1} \tag{8.102}$$

$$P_K(s) = \{P_{(K-2)}(s)\mu_1 + P_{(K-1)}(s)\mu_2\} \frac{1}{s + \sum_{i=1}^{2} \lambda_i} \tag{8.103}$$



for $K = 3, 6, 9, 12, \ldots, (n-2)$

$$P_{(n-1)}(s) = \frac{P_{(n-2)}(s)\lambda_1}{s} \tag{8.104}$$

$$P_n(s) = \frac{P_{(n-2)}(s)\lambda_2}{s} \tag{8.105}$$

To obtain state probabilities invert (8.100)–(8.105) to time domain [i.e., take inverse Laplace transforms of (8.100)–(8.105)]. The system operational availability, $A_s$, can be obtained from

$$A_s = \sum_{i=0}^{n-2} P_i(t) \tag{8.106}$$

for $i = 0, 3, 6, 9, \ldots, (n-2)$

### 8.5.8 *A* k*-out-of-*n *Three-State Device System with Common-Cause Failures*

This section presents a generalized Markov model to represent repairable $k$-out-of-$n$ units system with common-cause failures [14]. This mathematical model can also be applied to represent repairable series or parallel (two- or three-state device) network subject to common-cause failures. Some of the common-cause failures may occur due to (a) undetected design errors; (b) operator and maintenance errors; (c) common environments; (d) common manufacturer; (e) common energy source; (f) same repairman; or (g) equipment failure event—fire, flood, tornado, earthquake. A typical example may be a redundant configuration composed of two motorized fluid flow valves with common (control circuit) power supply. This type of situation is frequently encountered in power stations.

**Assumptions**

1. Three-state devices are identical.
2. The redundant system is only repaired when all devices fail in either failure modes (i.e., open, short, closed), or if the redundant system fails due to common-cause failures.
3. Common-cause failures can only occur if two or more three-state devices are present in a system.
4. A failed system is restored as good as new.
5. Common-cause and other failures are statistically independent.

**Notation**

$\lambda_i$ = constant open mode failure rate, for $i=0,1,2,3,\ldots,k$
$\alpha_i$ = constant short mode failure rate, for $i=0,1,2,3,\ldots,k$
$\gamma_i$ = constant common-cause failure rate, for $i=0,1,2,3,\ldots,(k-1)$
$\mu_{SH}$ = constant short failure mode repair rate
$\mu_o$ = constant open failure mode repair rate
$\mu_c$ = constant common-cause failure mode repair rate
$P_i(t)$ = state probability at time $t$ for $i=0,1,2,3,\ldots,n$
(Note: for $i=n$ represents open failure mode probability at time $t$)
$P_c(t)$ = common-cause failure mode probability at time $t$
$P_{SH}(t)$ = short failure mode probability at time $t$
$N$ = total number of devices in a system
$s$ = Laplace transform variable
$t$ = time

The associated equations with Figure 8.16 are

$$P_0'(t)=-(\lambda_0+\alpha_0+\gamma_0)P_0(t)+P_{SH}(t)\mu_{SH}+P_c(t)\mu_c+P_n(t)\mu_0 \tag{8.107}$$

$$P_1'(t)=-(\lambda_1+\alpha_1+\gamma_1)P_1(t)+P_0(t)\lambda_0 \tag{8.108}$$

$$P_2'(t)=-(\lambda_2+\alpha_2+\gamma_2)P_2(t)+P_1(t)\lambda_1 \tag{8.109}$$

$\vdots$

$$P_{k-1}'(t)=-(\lambda_{k-1}+\alpha_{k-1}+\gamma_{k-1})P_{k-1}(t)+P_{k-2}(t)\lambda_{k-2} \tag{8.110}$$

$\vdots$ for $k=2,3,4,\ldots,(n-1)$,

$$P_k'(t)=-(\lambda_k+\alpha_k)P_k(t)+P_{k-1}(t)\lambda_{k-1} \tag{8.111}$$

$\vdots$ for $k=(n-1)$,

$$P_n'(t)=-\mu_0P_n(t)+P_k(t)\lambda_k \tag{8.112}$$

$$P_{SH}'(t)=-\mu_{SH}P_{SH}(t)+\sum_{i=0}^{k}\alpha_iP_i(t) \qquad \text{for } k=n-1, \tag{8.113}$$

$$P_c'(t)=-\mu_cP_c(t)+\sum_{i=0}^{k-1}\gamma_iP_i(t) \qquad \text{for } k=n-1 \tag{8.114}$$

$$n=N \qquad \text{for } N\geqslant 2$$

$$\lambda_i=(N-i)\lambda \qquad \text{for } i=0,1,2,3,\ldots,N$$

At $P_0(0)=1$, other initial condition probabilities are equal to zero.



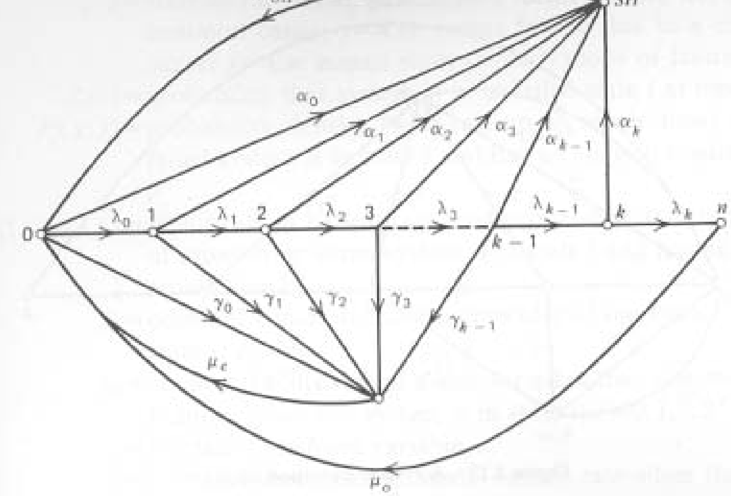


**Figure 8.16** Transition diagram.

The prime denotes differentiation with respect to time $t$. Laplace transforms of the state probability equations are

$$P_0(s)=\frac{1+P_{SH}(s)\mu_{SH}+P_c(s)\mu_c+P_n(s)\mu_0}{s+\lambda_0+\alpha_0+\gamma_0} \tag{8.115}$$

$$P_1(s)=\frac{P_0(s)\lambda_0}{s+\lambda_1+\alpha_1+\gamma_1} \tag{8.116}$$

$$P_2(s)=\frac{P_1(s)\lambda_1}{s+\lambda_2+\alpha_2+\gamma_2} \tag{8.117}$$

$\vdots$

$$P_{k-1}(s)=\frac{P_{k-2}(s)\lambda_{k-2}}{s+\lambda_{k-1}+\alpha_{k-1}+\gamma_{k-1}} \tag{8.118}$$

$\vdots$ for $k=2,3,4,\ldots,(n-1)$,

$$P_k(s)=\frac{P_{k-1}\lambda_{k-1}}{s+\lambda_k+\alpha_k} \qquad \text{for } k=(n-1), \tag{8.119}$$

$\vdots$

$$P_n(s)=\frac{P_k(s)\lambda_k}{s+\mu_0} \tag{8.120}$$

$$P_{SH}(s)=\frac{\sum_{i=0}^{k}\alpha_iP_i(s)}{s+\mu_{SH}} \qquad \text{for } k=n-1 \tag{8.121}$$

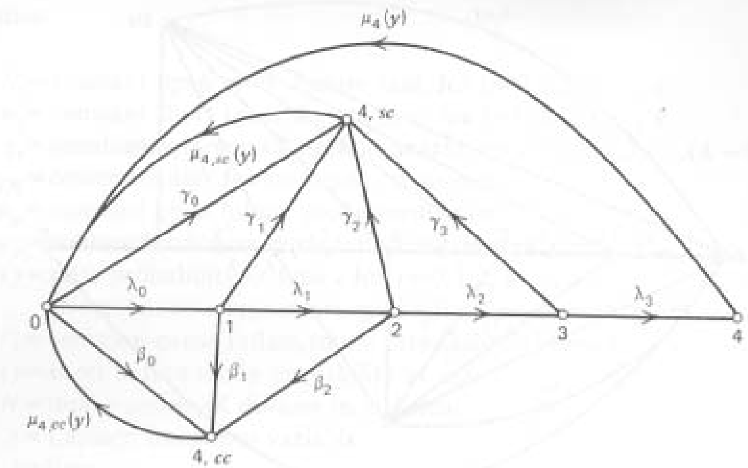


**Figure 8.17** A system transition diagram.

$$P_c(s) = \frac{\sum_{i=0}^{k-1} \gamma_i P_i(s)}{s+\mu_c} \quad \text{for} \quad k=n-1. \tag{8.122}$$

To use this model for series configuration interchange open failure mode probability with short (close) failure mode probability.

### 8.5.9 *A 4-Unit Redundant System with Common-Cause Failures*

This model [13] can be used for devices with two mutually exclusive failure modes and common-cause failures. The transition diagram of the model is shown in Figure 8.17.

**Assumptions**

1. Common-cause and other failures are *s*-independent.
2. Common-cause failures can only occur with more than one unit. 4-units are identical.
3. Units are repaired only when the system fails. A failed system is restored as good as new.
4. System repair times are arbitrarily distributed.

The transition diagram is shown in Figure 8.17.

**Notation**

$i$ = state of the unfailed system; number of failed units; $i$ = 0, 1, 2, 3



$j$ = state of the failed system; $j=4$ means failure not due to a common cause; $j=4,cc$ means failure due to a common-cause; $j=4,sc$ means short (closed) mode of failure

$P_i(t)$ = probability that system is in unfailed state $i$ at time $t$

$P_j(y,t)$ = probability density (with respect to repair time) that the failed system is in state $j$ and has an elapsed repair time of $y$

$\mu_j(y), q_j(y)$ = repair rate (a hazard rate) and probability density function of repair time when system is in state $j$ and has an elapsed repair time of $y$

$\beta_i$ = constant common-cause failure rate of the system when in state $i$; $\beta_3=0$

$\lambda_i$ = constant failure rate of a unit, for other than common-cause failures, when the system is in state $i$; $i=0,1,2,3$

$s$ = Laplace transform variable

$\gamma_i$ = constant short (closed mode) failure rate when the system is in state $i$; 0, 1, 2, 3

Equations (8.123)–(8.128) associated with Figure 8.17 are

$$\frac{dP_0(t)}{dt} + (\lambda_0+\beta_0+\gamma_0)P_0(t) = \int_0^\infty P_{4,cc}(y,t)\mu_{4,cc}(y)\,dy + \int_0^\infty P_4(y,t)\mu_4(y)\,dy + \int_0^\infty P_{4,sc}(y,t)\mu_{4,sc}(y)\,dy \tag{8.123}$$

$$\frac{dP_i(t)}{dt} + (\lambda_i+\beta_i+\gamma_i)P_i(t) - \lambda_{i-1}P_{i-1}(t) = 0 \tag{8.124}$$

for $i=1,2,3;\quad \beta_3=0$

$$\frac{\partial p_j(y,t)}{\partial t} + \frac{\partial p_j(y,t)}{\partial y} + \mu_j(y)p(y,t) = 0 \tag{8.125}$$

$$p_4(o,t) = \lambda_3 P_3(t) \tag{8.126}$$

$$p_{4,cc}(o,t) = P_o(t)\beta_o + P_1(t)\beta_1 + P_2(t)\beta_2 \tag{8.127}$$

$$p_{4,sc}(o,t) = P_o(t)\beta_o + P_1(t)\gamma_1 + P_2(t)\gamma_2 + P_3(t)\gamma_3 \tag{8.128}$$

$$\lambda_i = (i-1)\lambda$$

$$\gamma_i = (i-2)\gamma$$

$$P_i(o) = 1 \quad \text{for} \quad i=0 \quad \text{other } P_i(o)=0$$

$$P_j(y,0) = 0 \quad \text{for all } j$$

The Laplace transforms of the solution of (8.123)–(8.128)

$$P_0(s)=\left[s+\lambda_0+\beta_0+\gamma_0-\left(\beta_0+\frac{\beta_1}{A_1}+\frac{\beta_2}{A_2}\right)G_{4,cc}(s)\right.$$

$$\left.-\frac{G_4(s)\lambda_3}{A_3}-\left(\gamma_0+\frac{\gamma_1}{A_1}+\frac{\gamma_2}{A_2}+\frac{\gamma_3}{A_3}\right)G_{4,sc}\right]^{-1} \quad (8.129)$$

$$G_j(s)\equiv\int_0^\infty \exp(-sy)q_j(y)\,dy \qquad \text{for } j=4,4,cc, \text{ or } 4,sc$$

$$A_1\equiv\frac{s+\lambda_1+\beta_1+\gamma_1}{\lambda_0}$$

$$A_2\equiv\frac{A_1(s+\lambda_2+\beta_2+\gamma_2)}{\lambda_1} \qquad A_3\equiv\frac{A_2(s+\lambda_3+\gamma_3)}{\lambda_2}$$

$$P_i(s)=\frac{P_0(s)}{A_i} \qquad \text{for } i=1,2,3 \quad (8.130)$$

$$P_4(s)=\lambda_3P_3(s)\frac{1-G_4(s)}{s} \quad (8.131)$$

$$P_{4,cc}(s)=\left[\sum_{i=0}^{2}\beta_iP_i(s)\right]\frac{1-G_{4,cc}(s)}{s} \quad (8.132)$$

$$P_{4,sc}(s)=\left[\sum_{i=0}^{3}\gamma_iP_i(s)\right]\frac{1-G_{4,sc}(s)}{s} \quad (8.133)$$

To obtain time domain solutions, (8.129)–(8.133) can be transformed for given repair times distribution.

## 8.6 RELIABILITY OPTIMIZATION OF THREE-STATE DEVICE NETWORKS

This section deals with optimizing the number of redundant elements to obtain maximum reliability. Here, we focus on obtaining the optimum number of redundant elements for the series and parallel configuration only.



### 8.6.1 Series Network

Using expression 8.1 the series system reliability of identical elements is given by

$$R=(1-q_0)^n-q_s^n \quad (8.134)$$

To obtain optimum number of elements differentiate (8.134) with respect to $n$ and equate it to zero. The following results are obtained

$$\frac{\partial R}{\partial n}=\alpha_0^n\log_e\alpha_0-q_s^n\log_e q_s=0 \quad (8.135)$$

where $\alpha_0=(1-q_0)$.

Thus, rewriting (8.135) in terms of $n$ optimum number of elements $n^*$, we get

$$n^*=\frac{\log_e\left\{\dfrac{\log_e q_s}{\log_e\alpha_0}\right\}}{\log_e(\alpha_0/q_s)} \quad (8.136)$$

### 8.6.2 Parallel Network

The following expression is directly obtained from (8.136) by reasoning the duality of the series to parallel form

$$m^*=\frac{[\log_e(\log_e q_0/\log_e\alpha_s)]}{\log_e(\alpha_s/q_0)} \quad (8.137)$$

where $\alpha_s=(1-q_s)$ and $m^*$ is the optimum number of elements. Optimization of series or parallel network reliability subject to constraints is presented in reference 24.

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