CHAPTER 9

Appendices

Appendix I

Solution of simultaneous linear equations

Reliability evaluation often calls for the solution of a set of simultaneous linear equations of the form

$$AX = B$$

where A is a nonsingular coefficient matrix and X and B are column vectors. The results could be obtained using Cramer's rule which proceeds by evaluating determinants and expanding by minors. The system of n equations in n unknowns takes on the order of $n!$ multiplications. If $n = 25$ and each multiplication takes 10^{-6} seconds, the computation time required would be several million years. Several numerical methods have been devised to solve linear equations. This book covers only the basic principles of Gauss-Jordan method of elimination. Readers interested in further details should refer to one of the many excellent books available on numerical methods.

The basic procedure of this method is quite simple. The first variable from all but one of the equations is eliminated by adding an appropriate multiple of this equation to each of the others. The second variable is then eliminated from another equation in the same manner. The procedure continues until each of the equations has only one variable left. The result can then be read directly. This can be illustrated by solving the following set of linear equations.

$$x_1 + x_2 + 3x_3 = 4 \quad (1)$$

$$x_1 - x_2 + 4x_3 = 5 \quad (2)$$

$$2x_1 - x_2 + 3x_3 = 6 \quad (3)$$

Step 1 Remove x_1 from (2) and (3) by multiplying Equation (1) by (− 1) and (− 2) respectively and adding

$$x_1 + x_2 + 3x_3 = 4 \quad (4)$$

$$-2x_2 + x_3 = 1 \quad (5)$$

$$-3x_2 - 3x_3 = -2 \quad (6)$$

Step 2 Remove x_2 from (4) and (6) by multiplying (5) by ($\frac{1}{2}$) and ($-\frac{3}{2}$) respectively and adding to (4) and (6) respectively

$$x_1 + \tfrac{7}{2}x_3 = \tfrac{9}{2} \qquad (7)$$

$$-2x_2 + x_3 = 1 \qquad (8)$$

$$-\tfrac{9}{2}x_3 = -\tfrac{7}{2} \qquad (9)$$

Step 3 Remove x_3 from (7) and (8) by multiplying (9) by ($\frac{7}{9}$) and ($\frac{2}{9}$) and adding to (7) and (8) respectively

$$x_1 = \tfrac{16}{9}$$

$$-2x_2 = \tfrac{2}{9}$$

$$-\tfrac{9}{2}x_3 = -\tfrac{7}{2}$$

Step 4 The results can now be read

$$x_1 = \tfrac{16}{9}$$

$$x_2 = -\tfrac{1}{9}$$

$$x_3 = \tfrac{7}{9}$$

In the computer, the operations are done in matrix form. The initial step is to form an augmented array

$$[A \mid B]$$

This in our present example is of the form

$$\begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & -1 & 4 & 5 \\ 2 & -1 & 3 & 6 \end{bmatrix}$$

Row 1 is called the pivot row and the first element of this row is called the pivot element. The procedure consists in setting the elements above and below the pivot elements (diagonal elements) to zero.

First step. $R1$ (row 1) is the pivot row, a_{11} is the pivot element. Divide the pivot row by the pivot element. Set elements below the pivot element to zero by

$$R_2' = R_2 - a_{21}R1$$

$$R_3' = R_3 - a_{31}R1$$

$$\begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & -2 & 1 & 1 \\ 0 & -3 & -3 & -2 \end{bmatrix}$$

The prime indicates the resulting row.
Second Step. R_2 is the pivot row and a_{22} is the pivot element. Normalize the pivot row by dividing by the pivot element.

$$\begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -3 & -3 & -2 \end{bmatrix}$$

Set the elements above and below the pivot element by

$$R'1 = R1 - a_{12}R2$$

$$R'3 = R3 - a_{32}R2$$

$$\begin{bmatrix} 1 & 0 & \frac{7}{2} & \frac{9}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{9}{2} & -\frac{7}{2} \end{bmatrix}$$

Third Step. R_3 is the pivot row and a_{33} is the pivot element. Normalize the pivot row by dividing by the pivot element.

$$\begin{bmatrix} 1 & 0 & \frac{7}{2} & \frac{9}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & \frac{7}{9} \end{bmatrix}$$

Set the elements above the pivot row by

$$R'1 = R1 - a_{13}R3$$

$$R'2 = R2 - a_{23}R3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{16}{9} \\ 0 & 1 & 0 & -\frac{1}{9} \\ 0 & 0 & 1 & \frac{7}{9} \end{bmatrix}$$

The last column is the solution vector. It should be noted that special pivoting techniques are available for avoiding rounding off errors. For details of further refinements, books on numerical analysis should be consulted.

Appendix II

Shape of the hazard rate function of two series stage combinations in parallel

The expression of the probability density function of this stage combination as given in Chapter 6 is

$$f(x) = \omega_1 \rho_1 \frac{(\rho_1 x)^{a_1 - 1}}{(a_1 - 1)!} e^{-\rho_1 x} + \omega_2 \rho_2 \frac{(\rho_2 x)^{a_2 - 1}}{(a_2 - 1)!} e^{-\rho_2 x} \qquad (1)$$

The survivor function is

$$S(x) = \omega_1 \sum_{n=1}^{a_1} \frac{(\rho_1 x)^{n-1}}{(n-1)!} e^{-\rho_1 x} + \omega_2 \sum_{n=1}^{a_2} \frac{(\rho_2 x)^{n-1}}{(n-1)!} e^{-\rho_2 x} \qquad (2)$$

and the hazard rate function is

$$\phi(x) = \frac{f(x)}{S(x)}$$

(i) At the origin

$$\phi(0) = f(0)$$

since

$$S(0) = 1$$

The following conclusions can be drawn regarding the magnitude of the hazard rate at the origin

$$\phi(0) = \begin{cases} 0 & \text{if } a_1 > 1, \quad a_2 > 1 \\ \omega_1 \rho_1 & \text{if } a_1 = 1, \quad a_2 > 1 \\ \omega_2 \rho_2 & \text{if } a_1 > 1, \quad a_2 = 1 \\ \omega_1 \rho_1 + \omega_2 \rho_2 & \text{if } a_1 = a_2 = 1 \end{cases}$$

(ii) The derivative of $\phi(x)$ at the origin

$$\phi'(0) = \left. \frac{f'(x)S(x) + \{f(x)\}^2}{\{S(x)\}^2} \right|_{x=0}$$

$$= f'(0) + \{f(0)\}^2$$

Now by the initial value theorem

$$f'(0) = \lim_{s \to \infty} s[s\bar{f}(s) - f(0)]$$

The expression for $\bar{f}(s)$ is

$$\bar{f}(s) = \omega_1 \left(\frac{\rho_1}{\rho_1 + s} \right)^{a_1} + \omega_2 \left(\frac{\rho_2}{\rho_2 + s} \right)^{a_2}$$

The following conclusions can be drawn

(a) $a_1 = a_2 = 1$

Then

$$s\bar{f}(s) - f(0) = \frac{s\omega_1 \rho_1}{s + \rho_1} + \frac{s\omega_2 \rho_2}{s + \rho_2} - (\omega_1 \rho_1 + \omega_2 \rho_2)$$

$$= -\frac{\omega_1 \rho_1^2}{\rho_1 + s} - \frac{\omega_2 \rho_2^2}{\rho_2 + s}$$

Therefore

$$f'(0) = \lim_{s \to \infty} s \left[-\frac{\omega_1 \rho_1^2}{\rho_1 + s} - \frac{\omega_2 \rho_2^2}{\rho_2 + s} \right]$$

$$= -\omega_1 \rho_1^2 - \omega_2 \rho_2^2$$

and

$$\phi'(0) = -\omega_1 \rho_1^2 - \omega_2 \rho_2^2 + (\omega_1 \rho_1 + \omega_2 \rho_2)^2$$

$$= -\omega_1 \omega_2 (\rho_1 - \rho_2)^2$$

This expression is always negative and therefore for this condition, the hazard rate is always initially decreasing. See curve 1 of Fig. 6.9.

(b) $a_1 = 1$ and $a_2 = 2$

$$s\bar{f}(s) - f(0) = -\frac{\omega_1 \rho_1^2}{s + \rho_1} + \omega_2 s \left(\frac{\rho_2}{s + \rho_2} \right)^2$$

Therefore

$$f'(0) = \lim_{s \to \infty} s\{s\bar{f}(0) - f(0)\} = -\omega_1 \rho_1^2 + \omega_2 \rho_2^2$$

and

$$\phi'(0) = -\omega_1 \rho_1^2 + \omega_2 \rho_2^2 + (\omega_1 \rho_1)^2$$

$$= \omega_2 (\rho_2^2 - \omega_1 \rho_1^2)$$

The sign will depend upon that of the quantity inside the parenthesis. This is negative for curve 2 in Fig. 6.9 and therefore the hazard rate is initially decreasing.

(c) $a_1 = 2$ and $a_2 = 1$

$$\phi'(0) = \omega_1 (\rho_1^2 - \omega_2 \rho_2^2)$$

(d) $a_1 = 2$ and $a_2 = 2$

$$\phi'(0) = f'(0) \quad \text{since} \quad f(0) = 0$$

Therefore

$$\phi'(0) = \lim_{s\to\infty} s^2 \bar{f}(s)$$

$$= \omega_1\rho_1^2 + \omega_2\rho_2^2$$

The hazard rate is therefore initially increasing in this case. See curve 4 in Fig. 6.9

(e) $a_1 > 2, \quad a_2 > 2$

$$\phi'(0) = \lim_{s\to\infty} s^2 \bar{f}(s)$$

$$= 0$$

The hazard rate is, therefore, initially constant as can be verified from curve 5 of Fig. 6.9

(f) $a_1 = 1$ and $a_2 > 2$

$$s\bar{f}(s) - f(0) = -\frac{\omega_1\rho_1^2}{\rho_1 + s} + \omega_2 s\left(\frac{\rho_2}{s+\rho_2}\right)^{a_2}$$

$$\lim_{s\to\infty} s\{s\bar{f}(s) - f(0)\} = -\omega_1\rho_1^2$$

Therefore

$$\phi'(0) = -\omega_1\rho_1^2 + (\omega_1\rho_1)^2$$

$$= -\omega_1\omega_2\rho_1^2$$

(g) $a_1 > 2$ and $a_2 = 1$

$$\phi'(0) = -\omega_1\omega_2\rho_2^2$$

In cases (f) and (g) the hazard rate is, therefore, initially decreasing as can be seen from curve 3 of Fig. 6.9.

(III) $\phi(x)$ as $x \to \infty$

(a) $\rho_1 > \rho_2$

$$\lim_{x\to\infty} \phi(x) = \rho_2$$

(b) $\rho_2 > \rho_1$

$$\lim_{x\to\infty} \phi(x) = \rho_1$$

(c) $\rho_1 = \rho_2$

$$\lim_{x\to\infty} \phi(x) = \rho_1 = \rho_2$$

The limiting value of the hazard rate as x becomes large is therefore always the smaller of ρ_1 or ρ_2.

The three quantities

i. $\phi(x)$ as $x \to 0$

ii. $\phi'(x)$ as $x \to 0$

and

iii. $\phi(x)$ as $x \to \infty$

are enough to get an approximate idea of the shape of the hazard rate. A knowledge of the behaviour of these quantities is helpful in making finer adjustments in the shape of the hazard rate.

Appendix III

Hazard rate shape of series stages in series with a distinctive stage

(i) $\phi(x)$ at $x = 0$

It can be seen by examining the ratio of Equations (6.69) and (6.70) that

$$\phi(0) = 0$$

(ii) $\phi'(x)$ at $x = 0$

As in Appendix II

$$\phi'(0) = \lim_{s\to\infty} s[s\bar{f}(s) - f(0)] + \{f(0)\}^2$$

$$= \lim_{s\to\infty} s^2\bar{f}(s) \quad \text{since} \quad f(0) = 0$$

Now the Laplace transform of Equation (6.69) is

$$\bar{f}(s) = \left(\frac{\rho}{\rho+s}\right)^a \frac{1}{\rho_1 + s}$$

Therefore

$$\phi'(0) = \begin{cases} \rho\rho_1 & \text{if} \quad a = 1 \\ 0 & \text{if} \quad a > 1 \end{cases}$$

(iii) $\phi(x)$ as $x \to \infty$

It can be proved by examining the ratio of Equation (6.69) and (6.70) that

$$\lim_{x\to\infty} \phi(x) = \begin{cases} \rho_1 & \text{if} \quad \rho_1 < \rho \\ \rho & \text{if} \quad \rho < \rho_1 \end{cases}$$

The final value is therefore the lesser of the two.

Appendix IV

Series stages in series with two parallel stages

Derivation of expressions

Designating the state whose duration is represented by this combination as 0 and the state of not being in it as A the state transition diagram is shown in Fig. IV.1

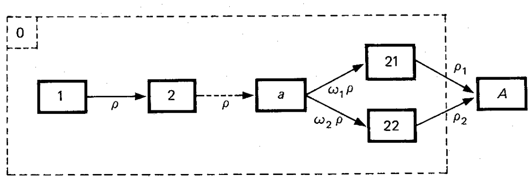

Fig. IV.1 State transition diagram to derive expressions for the stage combination

Assuming $p_1(0) = 1{\cdot}0$, the time spent in state 0 is identical with the time since the origin and as explained in Chapter 6.

$$f_0(x) = \rho_1 p_{21}(x) + \rho_2 p_{22}(x)$$

$$S_0(x) = \sum_{i=1}^{a} p_i(x) + p_{21}(x) + p_{22}(x)$$

and

$$\phi(x) = \frac{f_0(x)}{S_0(x)}$$

The differential equations for this system are

$$p_1'(t) = -\rho p_1(t)$$

$$p_2'(t) = \rho(p_1(t) - p_2(t))$$

$$\cdots$$

$$p_n'(t) = \rho(p_{n-1}(t) - p_n(t))$$

$$\cdots$$

$$p_a'(t) = \rho(p_{a-1}(t) - p_a(t))$$

$$p_{21}'(t) = \omega_1 \rho p_a(t) - p_{21}(t)\rho_1$$

$$p_{22}'(t) = \omega_2 \rho p_a(t) - p_{22}(t)\rho_2$$

Taking the Laplace transform of these differential equations

$$p_1(s) = \frac{1}{\rho + s} \tag{1}$$

$$p_2(s) = \frac{\rho}{(\rho + s)^2}$$

Similarly

$$p_n(s) = \frac{\rho^{n-1}}{(\rho + s)^n} \tag{2}$$

$$p_a(s) = \frac{\rho^{a-1}}{(\rho + s)^a} \tag{3}$$

$$p_{21}(s) = \omega_1 \left(\frac{\rho}{\rho + s}\right)^a \frac{1}{s + \rho_1} \tag{4}$$

and

$$p_{22}(s) = \omega_2 \left(\frac{\rho}{\rho + s}\right)^a \frac{1}{s + \rho_2} \tag{5}$$

The inverse of Expression (2) is

$$p_n(x) = \frac{(\rho x)^{n-1}}{(n-1)!} e^{-\rho x} \tag{6}$$

For taking the inverse of Expression (4), it can be expanded into partial fractions

$$p_{21}(s) = \omega_1 \left(\frac{\rho}{\rho + s}\right)^a \frac{1}{s + \rho_1} = \frac{N_1}{\rho + s} + \frac{N_2}{(\rho + s)^2} + \ldots$$

$$+ \frac{N_a}{(\rho + s)^a} + \frac{M}{\rho_1 + s}$$

The numerators can be determined as

$$M = (\rho_1 + s)p_{21}(s)|_{s=-\rho_1} = \omega_1 \left(\frac{\rho}{\rho - \rho_1}\right)^a$$

$$N_a = (\rho + s)^a p_{21}(s)|_{s=-\rho} = \omega_1 \rho^a \frac{1}{\rho - \rho_1}$$

and

$$N_{a-m} = \frac{1}{m!}\frac{d^m}{ds^m}(\rho + s)^a p_{21}(s)|_{s=-\rho} = (-1)^m \omega_1 \rho^a \frac{1}{(\rho_1 - \rho)^{m+1}}$$

Let

$$a - m = i$$

Then

$$N_i = (-1)^{a-i}\,\omega_1 \rho^a \frac{1}{(\rho_1 - \rho)^{a-i+1}}$$

Substituting these values into the expansion of $p_{21}(s)$ and inverting

$$p_{21}(x) = \omega_1 \left(\frac{\rho}{\rho - \rho_1}\right)^a e^{-\rho_1 x} - \omega_1 \rho^a \frac{1}{(\rho - \rho_1)^a} e^{-\rho x} - \ldots$$

$$- \omega_1 \rho^a \frac{1}{(\rho - \rho_1)^{a-i+1}} \frac{x^{i-1}}{(i-1)!} e^{-\rho x}$$

$$- \ldots - \omega_1 \rho^a \frac{1}{\rho - \rho_1} \frac{x^{a-1}}{(a-1)!} e^{-\rho x}$$

A similar expression can be now easily derived from $p_{22}(x)$ and finally

$$S_0(x) = p_0(x) = \sum_{n=1}^{a} \frac{(\rho x)^{n-1}}{(n-1)!} e^{-\rho x} + \omega_1 \left(\frac{\rho}{\rho - \rho_1}\right)^a$$

$$\times \left[e^{-\rho_1 x} - e^{-\rho x} \sum_{n=1}^{a} \frac{\{(\rho - \rho_1)x\}^{n-1}}{(n-1)!}\right]$$

$$+ \omega_2 \left(\frac{\rho}{\rho - \rho_2}\right)^a \left[e^{-\rho_2 x} - e^{-\rho x} \sum_{n=1}^{a} \frac{\{(\rho - \rho_2)x\}^{n-1}}{(n-1)!}\right] \qquad (7)$$

and

$$f_0(x) = p_{21}(x)\rho_1 + p_{22}(x)\rho_2$$

$$= \omega_1 \rho_1 \left(\frac{\rho}{\rho - \rho_1}\right)^a \left[e^{-\rho_1 x} - e^{-\rho x} \sum_{n=1}^{a} \frac{\{(\rho - \rho_1)x\}^{n-1}}{(n-1)!}\right]$$

$$+ \omega_2 \rho_2 \left(\frac{\rho}{\rho - \rho_2}\right)^a \left[e^{-\rho_2 x} - e^{-\rho x} \sum_{n=1}^{a} \frac{\{(\rho - \rho_2)x\}^{n-1}}{(n-1)!}\right] \qquad (8)$$

The Expression (8) could alternatively be derived using the fact that the Laplace transform of the sum of independent random variables is the product of their Laplace transforms. The probability density function of the random variable X representing the state 0 is the sum of the random variables X_1, X_2

denoting the states 1 to a and 21 and 22. The probability density function of stages 1 to a is

$$f_1(x) = \frac{\rho(\rho x)^{a-1}}{(a-1)!} e^{-\rho x}$$

and

$$\bar{f}_1(s) = \left(\frac{\rho}{\rho + s}\right)^a$$

The probability density function of the parallel stages is

$$f_2(x) = \omega_1 \rho_1\, e^{-\rho_1 x} + \omega_2 \rho_2\, e^{-\rho_2 x}$$

and the corresponding Laplace is

$$\bar{f}_2(s) = \omega_1 \frac{\rho_1}{\rho_1 + s} + \omega_2 \frac{\rho_2}{\rho_2 + s}$$

Now

$$\bar{f}_0(s) = \bar{f}_1(s) \cdot \bar{f}_2(s)$$

Substituting the values and inverting, $f_0(x)$ can be obtained. This is left as an exercise for the reader.

$$\text{The mean duration} = \text{The sum of the means}$$

$$= \frac{a}{\rho} + \frac{\omega_1}{\rho_1} + \frac{\omega_2}{\rho_2}$$

Variance

The two random variables X_1 and X_2 are independent and therefore the variance of X is the sum of the variances of X_1 and X_2.

$$\text{Variance} = \frac{a}{\rho^2} + \frac{2\omega_1}{\rho_1^2} + \frac{2\omega_2}{\rho_2^2} - \left(\frac{\omega_1}{\rho_1} + \frac{\omega_2}{\rho_2}\right)^2$$

$$= \frac{a}{\rho^2} + 2\left(\frac{\omega_1}{\rho_1^2} + \frac{\omega_2}{\rho_2^2}\right) - \left(\frac{\omega_1}{\rho_1} + \frac{\omega_2}{\rho_2}\right)^2$$

Shape Of The Hazard Rate

(i) $\phi(0) = \dfrac{f(0)}{S(0)} = 0$

(ii) $\phi'(0) = \lim_{s\to\infty} s^2 \bar{f}(s)$

Now

$$\bar{f}(s) = \left(\frac{\rho}{\rho+s}\right)^a \left(\omega_1 \frac{\rho_1}{\rho_1+s} + \omega_2 \frac{\rho_2}{\rho_2+s}\right)$$

$$\phi'(0) = \begin{cases} \omega_1\rho_1\rho + \omega_2\rho_2\rho & \text{if } a = 1 \\ 0 & \text{if } a > 1 \end{cases}$$

(iii) $\phi(x)$ as $x \to \infty$

(a) $\rho = \min(\rho, \rho_1, \rho_2), \quad \rho \neq \rho_1 \neq \rho_2$

$$\lim_{x\to\infty} \phi(x) = \lim_{x\to\infty} \frac{f(x)/x^{a-1}e^{-\rho x}}{S(x)/x^{a-1}e^{-\rho x}}$$

$$= \frac{\omega_1\rho_1\left(\frac{\rho}{\rho-\rho_1}\right)^a\left[-\frac{(\rho-\rho_1)^{a-1}}{(a-1)!}\right] + \omega_2\rho_2\left(\frac{\rho}{\rho-\rho_2}\right)^a\left[-\frac{(\rho-\rho_2)^{a-1}}{(a-1)!}\right]}{\frac{\rho^{a-1}}{(a-1)!} + \omega_1\left(\frac{\rho}{\rho-\rho_1}\right)^a\left[-\frac{(\rho-\rho_1)^{a-1}}{(a-1)!}\right] + \omega_2\left(\frac{\rho}{\rho-\rho_2}\right)^a\left[-\frac{(\rho-\rho_2)^{a-1}}{(a-1)!}\right]}$$

$$= \frac{\rho\rho_1\rho_2 - \rho^2(\omega_1\rho_1 + \omega_2\rho_2)}{\rho_1\rho_2 - \rho(\omega_1\rho_1 + \omega_2\rho_2)}$$

$$= \rho$$

(b) $\rho_1 = \min(\rho, \rho_1, \rho_2), \quad \rho \neq \rho_1 \neq \rho_2$

$$\lim_{x\to\infty} \phi(x) = \frac{f(x)/e^{-\rho_1 x}}{S(x)/e^{-\rho_1 x}}$$

$$= \frac{\omega_1\rho_1\left(\frac{\rho}{\rho-\rho_1}\right)^a}{\omega_1\left(\frac{\rho}{\rho-\rho_1}\right)^a}$$

$$= \rho_1$$

(c) $\rho_2 = \min(\rho, \rho_1, \rho_2), \quad \rho \neq \rho_1 \neq \rho_2$

$$\lim_{x\to\infty} \phi(x) = \rho_2$$

(d) $\rho = \rho_1 = \rho_2$, the combination becomes a Special Erlangian distribution having

$$\lim_{x\to\infty} \phi(x) = \rho$$

(e) $\rho_1 = \rho_2 < \rho$ then

$$\lim_{x\to\infty} \phi(x) = \frac{\omega_1\rho_1\left(\frac{\rho}{\rho-\rho_1}\right)^a + \omega_2\rho_2\left(\frac{\rho}{\rho-\rho_2}\right)^a}{\omega_1\left(\frac{\rho}{\rho-\rho_1}\right)^a + \omega_2\left(\frac{\rho}{\rho-\rho_2}\right)^a}$$

$$= \omega_1\rho_1 + \omega_2\rho_2$$

$$= \rho_1 = \rho_2$$

(f) $\rho = \rho_1 < \rho_2$

Then

$$\lim_{x\to\infty} \phi(x) = \rho$$

It can be concluded from above that

$$\lim_{x\to\infty} \phi(x) = \min(\rho, \rho_1, \rho_2)$$

Appendix V

Moments of Stage Combinations

Series of Identical Stages

The Laplace transform of the probability density function is

$$\bar{f}(s) = \left(\frac{\rho}{s+\rho}\right)^a$$

Differentiating successively and substituting $s = 0$

$$\bar{f}^{(r)}(0) = (-1)^r \frac{1}{\rho^r} \prod_{k=1}^{r} (a+k-1)$$

The rth moment is therefore

$$m_r = \frac{1}{\rho^r} \prod_{k=1}^{r} (a+k-1)$$

Two Series Stages in Parallel

The rth moment in this case can be written as

$$m_r = \frac{\omega_1}{\rho_1^r} \prod_{k=1}^{r} (a_1 + k - 1) + \frac{\omega_2}{\rho_2^r} \prod_{k=1}^{r} (a_2 + k - 1)$$

Series stages in series with a distinctive stage

$$\bar{f}(s) = \left(\frac{\rho}{s+\rho}\right)^a \left(\frac{1}{s+\rho_1}\right)$$

That is

$$(s+\rho)^a (s+\rho_1) \bar{f}(s) = \rho_1 \rho^a$$

Differentiating both sides

$$(s+\rho)(s+\rho_1)\bar{f}'(s) + \{a(s+\rho_1) + (s+\rho)\}\bar{f}(s) = 0 \tag{1}$$

Differentiating once again

$$(s+\rho)(s+\rho_1)\bar{f}''(s) + [2(s+\rho) + (a+1)(s+\rho_1)]\,\bar{f}'(s)$$
$$+ (a+1)\bar{f}(s) = 0 \tag{2}$$

Differentiating r times

$$(s+\rho)(s+\rho_1)\bar{f}^{(r)}(s) + \{r(s+\rho) + (a+r-1)(s+\rho_1)\}\bar{f}^{(r-1)}(s)$$
$$+ (r-1)(a+r-1)\bar{f}^{(r-2)}(s) = 0 \tag{3}$$

Putting

$$s = 0$$

and

$$\bar{f}^{(r)}(0) = (-1)^r m_r$$

From Equation (1)

$$\rho\rho_1 m_1 = a\rho_1 + \rho \quad \text{for} \quad r = 1$$

From Equation (2)

$$\rho\rho_1 m_2 - \{2\rho + (a+1)\rho_1\}m_1 = -(a+1) \quad \text{for} \quad r = 2$$

From Equation (3)

$$\rho\rho_1 m_r - \{r\rho + (a+r-1)\rho_1\}m_{r-1} + (r-1)(a+r-1)m_{r-2} = 0$$
$$\text{for} \quad r > 2$$

In the matrix form

$$[A]\,[m] = [B] \tag{4}$$

where

$A = [a_{ij}]$ is the coefficient matrix such that

$$a_{ij} = 0 \quad \text{if} \quad j > i \quad \text{or} \quad j < i-2$$
$$a_{ij} = \rho\rho_1 \quad \text{if} \quad i = j$$
$$a_{ij} = -\{i\rho + (a+i-1)\rho_1\} \quad \text{if} \quad j = i-1$$

and

$$a_{ij} = (i-1)(a+i-1) \quad \text{if} \quad j = i-2.$$

$$m = \begin{bmatrix} m_1 \\ m_2 \\ \cdot \\ \cdot \\ m_r \end{bmatrix}$$

$$B = \begin{bmatrix} \rho + a\rho_1 \\ -(a+1) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

The r moments can be found by solving the set of linear Equations (4).

Series stages in series with two parallel stages

This is equivalent to two 'series stages in series with a distinctive stage' in parallel. The rth moment for the whole combination is obtained by

$$[m] = [m_1] + [m_2]$$

$[m_1]$ and $[m_2]$ are found by equations,

$$[A_1]\,[m_1] = [B_1]$$

and

$$[A_2]\,[m_2] = [B_2]$$

In this case

$$[B_1] = \begin{bmatrix} (\rho + a\rho_1)\omega_1 \\ -(a+1)\omega_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

and

$$[B_2] = \begin{bmatrix} (\rho + a\rho_2)\omega_2 \\ -(a+1)\omega_2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Appendix VI

Calculation of the Jacobian Matrix for two series stages in parallel

Assuming a_1 and a_2, the remaining three parameters ρ_1, ρ_2 and ω_1 can be calculated by matching the first three moments. Therefore

$$X_0 = [\rho_{10}\rho_{20}\omega_{10}]^t, \quad \phi = [\phi_1\,\phi_2\,\phi_3]^t$$

The elements of the jth column of the Jacobian matrix can be obtained by differentiating

$$\phi_j = \frac{\omega_1}{\rho_1^j}\prod_{k=1}^{j}(a_1 + k - 1) + \frac{\omega_2}{\rho_2^j}\prod_{k=1}^{j}(a_2 + k - 1) - M_j$$

That is

$$\frac{\partial\phi_j}{\partial\rho_1} = -\frac{j\omega_1}{\rho_1^{j+1}}\prod_{k=1}^{j}(a_1 + k - 1)$$

$$\frac{\partial\phi_j}{\partial\rho_2} = -\frac{j\omega_2}{\rho_2^{j+1}}\prod_{k=1}^{j}(a_2 + k - 1)$$

and

$$\frac{\partial\phi_j}{\partial\omega_1} = \frac{1}{\rho_1^j}\prod_{k=1}^{j}(a_1 + k - 1) - \frac{1}{\rho_2^j}\prod_{k=1}^{j}(a_2 + k - 1)$$

Series Stages in Series with Two Parallel Stages

Assuming the number of stages to be a, the remaining four parameters, ρ_1, ρ_2, ρ and ω_1 can be calculated by matching the first four moments. The vector

$$\phi = m - M$$

where m and M are vectors of the stage model moments and the moments of the distribution to be approximated. Since $m = m_a + m_b$, the Jacobian of ϕ at X_0 becomes

$$\phi'(X_0) = m_a'(X_0) + m_b'(X_0)$$

$m_a'(X_0)$ and $m_b'(X_0)$ can be obtained by differentiating and solving

$$A_1 m_a = B_a \quad \text{and} \quad A_2 m_b = B_b$$

This solution can be obtained using Gauss elimination.