Power Management Circuits and Systems

I. Feedback Properties: Transfer function and sensitivity function

A non-inverted amplifier will be used as a testbed to verify the properties of passive feedback systems. Write KCL at non-inverting terminal.

\[ H(s) = \frac{A_{V}}{1 + A_{V} \left( \frac{R_{in}}{R_{in} + R_{F}} \right)} \]  

(1)

![Resistive feedback non-inverting amplifier configuration.](image)

Where \( A_{V} = \frac{V_{o}}{V_{+} - V_{-}} \) is the amplifier’s gain.
If we break the loop at amplifier inverting terminal, the term $A_V \left( \frac{R_{in}}{R_{in}+R_F} \right)$ corresponds to the system’s loop transfer function and the factor $\frac{R_{in}}{R_{in}+R_F}$ represents the feedback factor $\beta$.

The previous equation can be rewritten as

$$H(s) = \left( \frac{R_{in}+R_F}{R_{in}} \right) \left( \frac{1}{1+ \left( \frac{1}{A_V \left( \frac{R_{in}}{R_{in}+R_F} \right)} \right)} \right)$$

(2)

Although this expression looks more complex than the previous one, it is more practical since it is a bit more intuitive when designing systems.

Notice that in case the loop gain $A_V \left( \frac{R_{in}}{R_{in}+R_F} \right) \gg 1$, the system can safely approximated by the first factor since the second factor in eqn 2 is approximately unity, then we called this term as the ideal system transfer function.
\[ H_{\text{ideal}}(s) = \left( \frac{R_{\text{in}} + R_F}{R_{\text{in}}} \right) \]  

(3)

This is a very desirable case since the transfer function is function of the passive elements connected through the feedback systems; in fact the gain becomes equal to \(1/\beta\).

The overall (closed loop) transfer function is then “insensitive” to amplifier gain \((A_V)\) variations.

\(A_V\) is not well controlled and may varies by >100%

\(A_V\) is non-linear!!

Unfortunately amplifiers with infinite gain over the entire bandwidth of interest are not available; practical amplifier gain is usually modeled as

\[ A_V = \left( \frac{A_{V0}}{1 + \frac{s}{\omega_P}} \right) \]  

(4)

where \(A_{V0}\) is the amplifier DC gain and \(\omega_P\) is the amplifier’s pole. For a single dominant
pole system (other poles and zeros are placed far beyond this frequency), this pole defines the amplifier bandwidth. In equation 4, \( s (=j \omega) \) is the complex frequency variable.

For example, the -3dB frequency of the \( \mu A741 \) is only in the 5 Hz range while the open-loop DC gain is around \( 2 \cdot 10^5 \text{V/V} \).

The product of the open-loop DC gain and the bandwidth is defined as the OPAMP’s gain-bandwidth product (GBW). For the OPAMP \( \mu A741 \).

Fig. 2. Typical OPAMP open-loop magnitude response showing a single low-frequency pole and a roll-off gain of -20 dB/decade at medium and high frequencies.
According to (2) and (3), the gain error is determined by Eq. 5. If \( A_v0 \) is frequency-dependent, then by making use of Eq. 4, we can obtain the general form for the error function including the effects of the finite OPAMP bandwidth:

\[
\xi(s) = \frac{1}{A_v \left( \frac{R_{in}}{R_{in} + R_F} \right)}
\]

\[
\xi(s) = \left( 1 + \frac{s}{\omega_p} \right) \left( \frac{R_{in} + R_F}{R_{in}} \right). \tag{5}
\]

It is important to recognize that the error function is equal to \(1/(\text{loop gain})\).

Remember that loop gain is composed by the product of amplifier’s gain and feedback factor.
Fig. 3. OPAMP open-loop magnitude response and its effect on the error response for an inverting amplifier.

Fig. 3 illustrates the relationship between error function and OPAMP frequency response. The low-frequency error can be estimated from Eq. 5 (where \( s = 0 \)).

At the frequency of the OPAMP’s pole \( \omega_p \), the voltage gain decreases due to the presence of the pole. As a result, the error function increases as predicted by Eq. 5.

The higher the OPAMP bandwidth (\( \omega_p \)), the smaller the high frequency error is.

\[
\xi(s) = \left( 1 + \frac{s}{\omega_p} \right) \left( \frac{R_{in} + R_F}{A_{V0}} \right)
\]
Both limited DC gain and finite bandwidth of the amplifier increase the error function.

The rule of thumb is that for a closed loop system with a targeted error, due to finite amplifier’s parameters, smaller than $\zeta_0$, we must satisfy the following condition:

$$\left| \xi(s) \right| = \left( \sqrt{\frac{1 + \left( \frac{\omega_{\text{max}}}{\omega_p} \right)^2}{A_{V_0}}} \right) \left( \frac{R_m + R_F}{R_m} \right) \leq \xi_0 \cdot (6)$$

where $\omega_{\text{max}}$ is the maximum frequency of interest, often defined as system bandwidth. For the case $\omega_{\text{max}} / \omega_p > 10$, the condition for limited overall gain error can be simplified to

$$\left| \xi(s) \right| = \left( \frac{\omega_{\text{max}}}{\omega_p} \right) \left( \frac{R_m + R_F}{R_m} \right) \leq \xi_0 \cdot (7)$$
If $A_{V0} = 10^5 \text{V/V}$ and $R_F/R_{in} = 9$, the error measured at different frequencies.

<table>
<thead>
<tr>
<th>$1 + R_F/R_{in}$</th>
<th>$\omega / \omega_P$</th>
<th>Error magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.01</td>
<td>$\sim 10^{-4}$</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>$\sim 1.4 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>$\sim 10^{-3}$</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>$\sim 10^{-2}$</td>
</tr>
<tr>
<td>10</td>
<td>1000</td>
<td>$\sim 10^{-1}$</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>$\sim 10^{-2}$</td>
</tr>
<tr>
<td>1000</td>
<td>10</td>
<td>$\sim 10^{-1}$</td>
</tr>
<tr>
<td>10000</td>
<td>10</td>
<td>(\sim 10^0)</td>
</tr>
</tbody>
</table>
Sensitivity Function

When dealing with multivariable functions and some of the parameters may have significant variations then it is useful to compute the sensitivity functions to quantify the individual effects of system performance.

The sensitivity function of $H(x, y, z, \ldots)$ as function of the parameter $x$ is defined as

$$\int_{x}^{H} = \left(\frac{x}{H}\right) \left(\frac{dH}{dx}\right) = \frac{\left(\frac{dH}{dx}\right)}{\frac{H}{x}}$$

(8a)

The sensitivity function represents the first order variation of $H$ as function of the parameter $x$, normalized by the first order factor $H/x$. To get more insight on the relevance of this function, let us consider the following approximation:

$$\int_{x}^{H} = \frac{dH}{dx} \approx \frac{\Delta H}{\Delta x}$$

(8b)

where the first derivative is expressed by a discrete approximation $\Delta H/\Delta x$. The sensitivity
function then measures the normalized variation of the transfer function \( \frac{\Delta H}{H} \) as function of the normalized variation of parameter \( \frac{\Delta x}{x} \).

Let us represent the normalized error (in %) of H as function of the variation in x; then

\[
\frac{\Delta H}{H} = \left( \int_{x}^{H} \right) \frac{\Delta x}{x}
\]

For instance if the value of the sensitivity function is 10, then 1% variation in parameter x will produce a variation of 10% in the overall transfer function H.

It is highly desirable to maintaining the sensitivity function of H with respect to critical parameters lesser than 1. More examples will be consider later.
Non-Inverting Amplifier. Let us consider the case of the equation 2, rewritten as follows:

\[
H(s) = \left(\frac{R_{in}+R_F}{R_{in}}\right)\left(\frac{1}{1+\frac{1}{AV\left(\frac{R_{in}}{R_{in}+R_F}\right)}}\right) = \left(\frac{R_{in}+R_F}{R_{in}}\right)\left(\frac{1}{1+\xi}\right)
\]  

(2b)

The computation of the sensitivity of \(H\) with respect to \(\xi\) yields,

\[
\int_{\xi}^{H} \frac{d}{d\xi} \left(\frac{R_{in}+R_F}{R_{in}}\right)\left(\frac{1}{1+\xi}\right) = \left(\frac{\xi}{1+\xi}\right)\left(\frac{1}{1+\xi}\right)
\]

\[
= \left(\xi(1+\xi)\right)\left(-\left(\frac{1}{1+\xi}\right)^2\right) = -\frac{\xi}{1+\xi}
\]  

(9)

If, for instance, we expect to see variations of \(\xi\) in the range of 100% but we do want to reduce those effects on \(H(s)\) to be no more than 1%, then \(\xi\) must be maintained under 0.01. In case \(\xi \ll 1\), equation 9 can be approximated as
\[
\int_{\xi}^{H} \cong -\xi
\]

where

\[
\zeta = \left( \frac{\omega_{\max}}{\omega_{p}} \right) \left( \frac{R_{\text{in}} + R_{F}}{R_{\text{in}}} \right) = \left( \frac{\omega_{\max}}{GBW} \right) \left( \frac{R_{\text{in}} + R_{F}}{R_{\text{in}}} \right); \text{ GBW is the gain-bandwidth product of the amplifier.}
\]
**Inverting Amplifier.** The case of the inverting amplifier is, surprisingly, a bit more complex than the case of the non-inverting amplifier.

\[
V_- = \left(\frac{R_F}{R_{in}+R_F}\right)V_i + \left(\frac{R_{in}}{R_{in}+R_F}\right)V_0 \tag{10}
\]

\[
V_0 = -A_V V_- = -A_V \left(\frac{R_F}{R_{in}+R_F}\right)V_i - A_V \left(\frac{R_{in}}{R_{in}+R_F}\right)V_0 \tag{11}
\]

The first term of the right most term is the so-called open loop gain and can be obtained breaking the feedback loop at the amplifier’s input and grounding the right most terminal of \(R_F\).

A voltage divider results in cascade with the inverting amplifier gain. The second right hand most term is our old friend: the (open) loop gain which is obtained in this case by breaking the loop and grounding \(V_i\) terminal.
Solving equation 11 leads to the desired transfer function; after some mathematical manipulations we can find that

\[
\frac{V_0}{V_i} = - \left( \frac{R_F}{R_{in}} \right) \left( \frac{1}{1 + \frac{1}{AV \left( \frac{R_{in}}{R_{in} + R_F} \right)}} \right)
\]

Fortunately the error function is the same as for the case of the non-inverting amplifier and we do not have to repeat the analysis.

Fig. 4. Resistive feedback inverting amplifier configuration.
Voltage Controlled Current Amplifiers with floating elements in feedback. The analysis of this type of amplifiers is cumbersome and often we get lost on the algebra. Let us first consider the case of the inverting amplifier including amplifier’s input and output impedance as depicted below. The amplifier’s small signal model employing a voltage controlled current source is depicted in Fig. 3.35b.

![Fig. 3.35a) Inverting amplifier with OPAMP input impedance Z_in and load impedance Z_L, and b) its small-signal equivalent circuit where R_o = ∞.](image)

Using KCL at both nodes Vx and Vo we find
\[ V_x \left( \frac{1}{Z_{in}} + \frac{1}{Z_1} + \frac{1}{Z_F} \right) - \frac{V_0}{Z_F} = \frac{V_i}{Z_{in}} \quad (11) \]

\[ V_x \left( g_m - \frac{1}{Z_F} \right) + V_0 \left( \frac{1}{Z_L} + \frac{1}{Z_F} \right) = 0 \quad (12) \]

These equations can be solved for both \( V_0 \) and \( V_x \), as function of the input signal \( V_{in} \).

A more elegant yet more insightful solution consists on the solution of the network employing the Mason rule.

Although it is not demonstrated here, it is true that the transfer function of a given system represented by “unidirectional building blocks” can always be obtained by employing Mason’s rule.
i) Unidirectional building blocks means that the output is driven by the input, but variations at the output do not affect at all the input voltage. Information flows in one direction: input $\rightarrow$ output!
   a. Examples of these blocks are the voltage controlled voltage sources, voltage controlled current sources, current controlled voltage sources and current controlled current sources.
   b. Examples of non-directional elements are the transformers, and floating impedances (resistors, capacitors, inductors and combination of these elements, amplifiers with feedback elements).

Once the system is represented by unidirectional elements, we must:
   ii) Identify the direct trajectories(paths) from the input(s) to output
   iii) The loops must also be identified
   iv) Loops that are not touched by certain direct paths
   v) The loops that are not sharing elements or nodes (un-touched loops)
Some of the definitions used above are highlighted in the following examples. Assuming that the building blocks are unidirectional, we can identify the following “direct paths:

i) \( Ky \cdot An(s) \cdotVy \)

ii) \( Am(s) \cdot An(s) \cdot Vx \)

We can also identify one loop

iii) \( Am(s) \)

loop is not touched by \( Vy \) path
For the schematic, the following paths and loop can be identified:

i) Direct path 1 from Vy: $K_yA_Ny$

ii) Direct path 2 from Vx: $A_m(s)A_n(s)V_x$

iii) Loop 1: $A_m(s)$

iv) Loop 2: $A_n(s)$

v) Loop 3: $A_m(s)A_n(s)$

vi) Notice that $A_m(s)$ and $A_n(s)$ do not show any element in common. **Non-touching loops**
MASON Rule: Once the direct paths and loops are identify, system transfer function can be easily obtained according to Mason’s rule:

If the system is linear, then every single input generates an output component that can be computed according to the following rule

\[
\frac{v_o}{v_i} = \sum \text{direct paths} - \sum \text{product of direct paths and untouched loops} + \ldots . \frac{1}{1 - \sum \text{loops} + \sum \text{product of untouched loops} - \ldots} \tag{12}
\]

In the case of the first diagram we can obtain the following transfer functions:

\[
H_{ox} = \frac{V_0}{V_x} = \frac{A_m(s)A_n(s)}{1-A_m(s)} \tag{12}
\]

and

\[
H_{oy} = \frac{V_0}{V_y} = \frac{K_yA_n(s)}{1} \tag{12}
\]

Notice that \(V_yA_n(s)\) does not touch the loop \(A_m(s)\), but this loop does not have any contribution when \(V_x=0\).
Once these transfer functions are obtained, the computation of the overall output voltage is computed as follows:

\[ V_0 = H_{oy} V_y + H_{ox} V_x \]  \hspace{1cm} (12)

For the case of the second schematic, the individual transfer functions can be computed as

\[ H_{ox} = \frac{V_0}{V_x} = \frac{A_m(s)A_n(s)}{1-A_m(s)-A_n(s)-A_n(s)A_m(s)-A_n(s)A_m(s)} \]  \hspace{1cm} (12)

The denominator is the result of the single loop \( A_m(s) \), the single loop \( A_n(s) \), the big loop involving \( A_m(s)A_n(s) \) and the last term is due to the product of the two non-touching loops \( A_n(s)A_m(s) \).

\[ H_{oy} = \frac{V_0}{V_y} = \frac{K_y A_n(s)-\{K_y A_n(s)\}[A_m(s)]}{1-A_m(s)-A_n(s)-A_n(s)A_m(s)-A_n(s)A_m(s)} \]  \hspace{1cm} (12)
Floating Impedances: Limiting elements for Mason’s rule.

A major issue is the mapping of circuits that have bi-directional elements connecting different nodes.

Will continue!