I. Feedback Properties: Transfer function and sensitivity function

In the first part of this section, the following non-inverted amplifier will be used as a testbed to verify the properties of passive feedback systems. Please verify that amplifier transfer function is given by the following expression

\[ H(s) = \frac{A_V}{1 + A_V \left( \frac{R_{in}}{R_{in} + R_F} \right)} \]  

(1)

Where \( A_V = \frac{V_o}{V_i - V_o} \) is the amplifier’s gain. Notice that if we break the loop at amplifier inverting terminal, the term \( A_V \left( \frac{R_{in}}{R_{in} + R_F} \right) \) corresponds to the system’s loop transfer function and the factor \( \frac{R_{in}}{R_{in} + R_F} \) represents the feedback factor \( \beta \). The previous equation can be rewritten as

\[ H(s) = \left( \frac{R_{in} + R_F}{R_{in}} \right) \left( \frac{1}{1 + A_V \left( \frac{R_{in}}{R_{in} + R_F} \right)} \right) \]  

(2)

Although this expression looks more complex than the previous one, it is more practical since it is a bit more intuitive when designing systems. Notice that in case the loop gain \( A_V \left( \frac{R_{in}}{R_{in} + R_F} \right) \gg 1 \), the system can safely approximated by the first factor since the second factor in eqn 2 is approximately unity, then we called this term as the ideal system transfer function

\[ H_{ideal}(s) = \left( \frac{R_{in} + R_F}{R_{in}} \right) \]  

(3)

This is a very desirable case since the transfer function is function of the passive elements connected.
through the feedback systems; in fact the gain becomes equal to $1/\beta$. The overall (closed loop) transfer function is then “insensitive” to amplifier gain ($A_V$) variations. Unfortunately this amplifiers with infinite gain over the entire bandwidth of interest are not available; practical amplifier gain is usually modeled as

$$A_V = \left( \frac{A_{V0}}{1 + \frac{s}{\omega_P}} \right)$$

where $A_{V0}$ is the amplifier DC gain and $\omega_P$ is the amplifier’s pole. For a single dominant pole system (other poles and zeros are placed far beyond this frequency), this pole defines the amplifier bandwidth. In equation 4, $s (=j\omega)$ is complex frequency variable.

For example, the -3dB frequency of the μA741 is only in the 10 Hz range while the open-loop DC gain is around $2\times10^5 \text{V/V}$. The product of the open-loop DC gain and the bandwidth is defined as the OPAMP’s gain-bandwidth product (GBW). For the OPAMP μA741, the typical value of GBW is around 1.5 MHz. These parameters and the transfer function of the μA741 are illustrated in the following Figure.

![Fig. 2. Typical OPAMP open-loop magnitude response showing a single low-frequency pole and a roll-off gain of -20 dB/decade at medium and high frequencies.](image)

Let us define the transfer function error introduced if the open-loop gain of the amplifier is finite. According to (2) and (3), the gain error is determined by Eq. 5. If $A_{v0}$ is frequency-dependent, then by making use of Eq. 4, we can obtain the general form for the error function including the effects of the finite OPAMP bandwidth:

$$\xi(s) = \left( \frac{S}{\omega_P} \right) \left( \frac{R_{in} + R_F}{A_{V0}} \right).$$

It is important to recognize that the error function is equal to $1/(\text{loop gain})$. Remember that loop gain is composed by the product of amplifier’s gain and feedback factor.
Fig. 3. OPAMP open-loop magnitude response and its effect on the error response for an inverting amplifier.

Based on the above expression, the pole of the OPAMP leads to a zero in the error transfer function. Fig. 3 illustrates the relationship between error function and OPAMP frequency response. The low-frequency error can be estimated from Eq. 5 (where \( s = 0 \)). At the frequency of the OPAMP’s pole \( \omega_p \), the voltage gain decreases due to the presence of the pole. As a result, the error function increases as predicted by Eq. 5. In a first-order approximation, the error function increases by a factor of 10 when the frequency increases by the same factor after the location of the pole. The higher the OPAMP bandwidth (\( \omega_p \)), the smaller the high frequency error is.

Both limited DC gain and finite bandwidth of the amplifier increase the error function. For the case of the monotonic amplifier’s transfer function (eqn 4), the maximum error occurs at the edge of targeted closed loop transfer function. The rule of thumb is that for a closed loop system with a targeted error, due to finite amplifier’s parameters, smaller than \( \zeta_0 \), we must satisfy the following condition:

\[
|\xi(s)| = \left| \frac{1}{A_{v0}} \left[ 1 + \frac{\omega_{max}}{\omega_p} \right]^2 \right| \left( \frac{R_m + R_F}{R_m} \right) \leq \zeta_0. \tag{6}
\]

where \( \omega_{max} \) is the maximum frequency of interest, often defined as system bandwidth. For the case \( \omega_{max} / \omega_p > 10 \), the condition for limited overall gain error can be simplified to
Let us consider some practical cases to get more insight on these issues due to limited performance in the operational amplifier.

If $A_{V0} = 10^5 \text{V/V}$ and $R_f/R_{in} = 9$, the error measured and different frequencies is listed in Table 1. Notice in the first 5 rows that the error increases monotonically with frequency. This is a result of the limited bandwidth of the OPAMP. We have to remember that the open-loop voltage gain reduces approximately proportionally to frequency after $\omega_p$. The desired amplifier gain also affects the error magnitude function; these effects are visible when comparing the results in rows with $\omega/\omega_p=10$ and different desirable voltage gain.

Table 3.2 Closed-loop gain magnitude error: example amplifier for different ideal gain factors $1+R_f/R_{in}$ and $\omega/\omega_p$ ratios.

<table>
<thead>
<tr>
<th>$1+R_f/R_{in}$</th>
<th>$\omega / \omega_p$</th>
<th>Error magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.01</td>
<td>$\sim 10^{-4}$</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>$\sim 1.4 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>$\sim 10^{-3}$</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>$\sim 10^{-2}$</td>
</tr>
<tr>
<td>10</td>
<td>1000</td>
<td>$\sim 10^{-1}$</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>$\sim 10^{-2}$</td>
</tr>
<tr>
<td>1000</td>
<td>10</td>
<td>$\sim 10^{-1}$</td>
</tr>
<tr>
<td>10000</td>
<td>10</td>
<td>$\sim 10^0$</td>
</tr>
</tbody>
</table>

**Sensitivity Function**
When dealing with multivariable functions and some of the parameters may have significant variations then it is useful to compute the sensitivity functions to quantify the individual effects of system performance. The sensitivity function of $H(x, y, z, \ldots)$ as function of the parameter $x$ is defined as

$$
\int_x^H = \left( \frac{x}{H} \right) \left( \frac{dH}{dx} \right) = \left( \frac{dH}{H} \right) \frac{x}{x}
$$

(8a)

The sensitivity function is formally obtained represents the first order variation of $H$ as function of the parameter $x$, normalized by the first order factor $H/x$. To get more insight on the relevance of this function, let us consider the following approximation:

$$
\int_x^H \approx \frac{\Delta H}{\Delta x}
$$

(8b)

where the first derivative is expressed by a discrete approximation $\Delta H/\Delta x$. The sensitivity function then measures the normalized variation of the transfer function $\Delta H/H$ as function of the normalized variation of parameter $\Delta x/x$. For instance if the value of the sensitivity function in 8 is 10, then 1% variation in parameter $x$ will produce a variation of 10% in the overall transfer function $H$. It is highly desirable to maintaining the sensitivity function of $H$ with respect to critical parameters lesser than 1. For the case of large parameter variations such as gain and bandwidth of the operational amplifier it is highly desirable to keep the sensitivity functions well below 1.

**Main concept behind Feedback.**

**Non-Inverting Amplifier.** Let us consider the case of the equation 2, rewritten as follows:

$$
H(s) = \left( \frac{R_{in}+R_E}{R_{in}} \right) \left( \frac{1}{1+\frac{1}{AV(\frac{R_{in}}{R_{in}+R_E})}} \right) = \left( \frac{R_{in}+R_E}{R_{in}} \right) \left( \frac{1}{1+\xi} \right)
$$

(2b)

The computation of the sensitivity of $H$ with respect to $\xi$ yields,

$$
\int_x^H = \left( \frac{\xi}{\left( \frac{R_{in}+R_E}{R_{in}} \right) \left( \frac{1}{1+\xi} \right)} \right) \left( \frac{d}{d\xi} \left( \frac{R_{in}+R_E}{R_{in}} \right) \left( \frac{1}{1+\xi} \right) \right) = \left( \frac{\xi}{1+\xi} \right) \left( \frac{d}{d\xi} \left( \frac{1}{1+\xi} \right) \right)
$$

$$
= \left( \xi (1 + \xi) \right) \left[ -\left( \frac{1}{1+\xi} \right)^2 \right] = -\frac{\xi}{1+\xi}
$$

(9)

If, for instance, we expect to see variations of $\xi$ in the range of 100% but we do want to reduce those effects on $H(s)$ to be no more than 1%, then $\xi$ must be maintained under 0.01. Some properties of the sensitivity
function are included in appendix A to facilitate its computation. In case $\xi \ll 1$, equation 9 can be approximated as

$$\int_{x}^{H} \approx -\xi$$

(9)

where $\zeta = \left( \frac{\omega_{\text{max}}}{\omega_{p}} \right) \left( \frac{R_{\text{in}} + R_{F}}{R_{\text{in}}} \right) = \left( \frac{\omega_{\text{max}}}{\text{GBW}} \right) \left( \frac{R_{\text{in}} + R_{F}}{R_{\text{in}}} \right)$: GBW is the gain-bandwidth product of the amplifier.

**Inverting Amplifier.** The case of the inverting amplifier is, surprisingly, a bit more complex than the case of the non-inverting amplifier. There are two fundamental reasons for this complications: the voltage at the amplifier’s inverting terminal is determined by the combination of $v_i$ and $v_o$, leading to the following results:

$$V_{-} = \left( \frac{R_{F}}{R_{\text{in}} + R_{F}} \right) V_{i} + \left( \frac{R_{\text{in}}}{R_{\text{in}} + R_{F}} \right) V_{0}$$

(10)

$$V_{0} = -A_{V} V_{-} = -A_{V} \left( \frac{R_{F}}{R_{\text{in}} + R_{F}} \right) V_{i} - A_{V} \left( \frac{R_{\text{in}}}{R_{\text{in}} + R_{F}} \right) V_{0}$$

(11)

The first term of the right most term is the so-called open loop gain and can be obtained breaking the feedback factor at the output and grounding the right most terminal of $R_{F}$. A voltage divider results in cascade with the inverting amplifier gain. The second right hand most term is our old friend: the (open) loop gain which is obtained in this case by breaking the loop and grounding $V_i$ terminal.

Solving equation 11 leads to the desired transfer function; after some mathematical manipulations we can find that

$$V_{0} = -\left( \frac{R_{F}}{R_{\text{in}}} \right) \left( \frac{1}{1 + \frac{1}{A_{V} \left( \frac{R_{\text{in}}}{R_{\text{in}} + R_{F}} \right)}} \right)$$

(12)

Fortunately the error function is the same as for the case of the non-inverting amplifier and we do not have to repeat the analysis.
Voltage Controlled Current Amplifiers with floating elements in feedback. The analysis of this type of amplifiers is cumbersome and often we get lost on the algebra. The results are not evident and hard to be properly interpreted; more relevant is that we get very little insight and so guidelines to optimize the system design process. Let us first consider the case of the inverting amplifier including amplifier’s input and output impedance as depicted in Fig. xx. The amplifier’s small signal model employing a voltage controlled current source is depicted in Fig. xxb.

Using KCL at both nodes we find

\[ V_X \left( \frac{1}{Z_{in}} + \frac{1}{Z_1} + \frac{1}{Z_F} \right) - \frac{V_0}{Z_F} = \frac{V_i}{Z_{in}} \]  \hspace{1cm} (11)

\[ V_X \left( g_m - \frac{1}{Z_F} \right) + V_0 \left( \frac{1}{Z_L} + \frac{1}{Z_F} \right) = 0 \]  \hspace{1cm} (11)

A more elegant yet more insightful solution consists on the solution of the network employing the Mason rule. Although it is not demonstrated here, it is true that the transfer function of a given system represented by “unidirectional building blocks” can always be obtained by identifying

i) Unidirectional building blocks means that the output is driven by the input, but variations at the output does not affect at all the block’s input.

a. Examples of these blocks are the voltage controlled voltage sources, voltage controlled current sources, current controlled voltage sources and current controlled current sources.

b. Examples of non-directional elements are the transformers, and floating impedances
Once the system is represented by unidirectional elements, we must:

ii) Identify the direct trajectories(paths) from the input(s) to output

iii) The loops must also be identified

iv) Loops that are not touched by certain direct paths

v) The loops that are not sharing elements or nodes (un-touched loops)

Some of the definitions used above are highlighted in the following examples. Assuming that the building blocks are unidirectional, we can identify the following “direct paths:

i) $KyAn(s)Vy$

ii) $Am(s)An(s)Vx$

We can also identify one loop

iii) $Am(s)$

For the schematic, the following paths and loop can be identified:

i) Direct path 1 from $Vy$: $KyAnVy$
ii) Direct path 2 from $V_x > Am(s) \text{ An}(s)V_x$

iii) Loop 1: $Am(s)$

iv) Loop 2: $An(s)$

v) Loop 3: $Am(s) \text{An}(s)$

vi) Notice that $Am(s)$ and $An(s)$ Do not show any element in common.

**MASON Rule:** Once the direct paths and loops are identify, system transfer function can be easily obtained according to Mason’s rule as follows:

If the system is linear, then every single input generates an output component that can be computed according to the following rule

\[
\frac{v_o}{v_i} = \sum \text{ direct paths} - \sum \text{ product of direct paths and untouched loops} + \ldots.
\]

\[
1 - \sum \text{ loops} + \sum \text{ product of untouched loops} - \ldots
\]

(12)

In the case of the first diagram we can obtain the following transfer functions:

\[
H_{ox} = \frac{V_o}{V_x} = \frac{A_m(s)A_n(s)}{1-A_m(s)}
\]

and

\[
H_{oy} = \frac{V_o}{V_y} = \frac{K_yA_n(s)}{1}
\]

(12)

Notice that $V_y An(s)$ does not touch the loop $Am(s)$. Once these transfer functions are obtained, the computation of the overall output voltage is computed as follows:

\[
V_0 = H_{oy}V_y + H_{ox}V_x
\]

(12)

For the case of the second schematic, the individual transfer functions can be computed as

\[
H_{ox} = \frac{V_o}{V_x} = \frac{A_m(s)A_n(s)}{1-A_m(s)-A_n(s)-A_m(s)A_n(s)+A_n(s)A_m(s)}
\]

(12)

The denominator is the result of the single loop $Am(s)$, the single loop $An(s)$, the big loop involving $Am(s)An(s)$ and the last term is due to the product of the two non-touching loops $An(s)Am(s)$.

\[
H_{oy} = \frac{V_o}{V_y} = \frac{K_yA_n(s) - [K_yA_n(s)](1-A_m(s))}{1-A_m(s)-A_n(s)-A_m(s)A_n(s)+A_n(s)A_m(s)}
\]

(12)

The overall voltage gain is given by equation xx. Other examples will be discussed in class.
Floating Impedances. A major issue is the mapping of circuits that have bi-directional elements connecting different nodes. Let us consider the following generic diagram involving a transfer function $H(s)$ and an admittance in feedback; see fig. xa. It is not difficult to demonstrate through nodal equations that the representation of the floating element through the four components depicted in fig. xb. The importance of this representation is that the properties of the passive feedback became apparent in this equivalent diagram.

i) The input impedance of the architecture is obtained as

$$Y_{in} = Y_F + \left[\frac{V_0}{V_{in}}\right] Q Y_F = Y_F \left(1 + \left[\frac{V_0}{V_{in}}\right] Q\right)$$

ii) The amplifier output impedance can be computed as follows

$$Y_{out} = Y_F + \left[\frac{V_0}{V_{in}}\right] Q Y_F = Y_F \left(1 + \left[\frac{V_0}{V_{in}}\right] Q\right)$$

iii) Will continue!