Power computations
Power and Energy

- **Instantaneous power** $p(t)$ is defined as the product of instantaneous voltage $v(t)$ and instantaneous current $i(t)$.

$$p(t) = v(t)i(t)$$

- **Average power** $P$ is the time average of $p(t)$ over one or more periods

$$P = \frac{1}{T} \int_{t_0}^{t_0+T} p(t) \, dt$$

$T$ is the period of the power waveform
Power and Energy

• Energy, or work, is the integral of instantaneous power

\[ W = \int_{t_1}^{t_2} p(t) \, dt \]

• Power has units of watts and energy has units of joules.
• Power can also be computed from energy per period

\[ P = \frac{W}{T} \]
Power and Energy

Example

\[ v(t) = \begin{cases} 
20 \text{ V} & 0 < t < 10 \text{ ms} \\
0 & 10 \text{ ms} < t < 20 \text{ ms} 
\end{cases} \]

\[ i(t) = \begin{cases} 
20 \text{ V} & 0 < t < 6 \text{ ms} \\
-15 \text{ A} & 6 \text{ ms} < t < 20 \text{ ms} 
\end{cases} \]

\[ p(t) = \begin{cases} 
400 \text{ W} & 0 < t < 6 \text{ ms} \\
-300 \text{ W} & 6 \text{ ms} < t < 10 \text{ ms} \\
0 & 10 \text{ ms} < t < 20 \text{ ms} 
\end{cases} \]
Power and Energy

Example contd

• Energy absorbed by the device in one period

\[
W = \int_0^T p(t) \, dt = \int_0^{0.006} 400 \, dt + \int_0^{0.010} (-300) \, dt + \int_0^{0.020} 0 \, dt = 2.4 - 1.2 = 1.2 \text{ J}
\]

• Average power is

\[
P = \frac{1}{T} \int_0^T p(t) \, dt = \frac{1}{0.020} \left( \int_0^{0.006} 400 \, dt + \int_0^{0.010} (-300) \, dt + \int_0^{0.020} 0 \, dt \right) = 60W
\]
Power and Energy

• For applications such as battery charging circuits and DC power supplies, average power absorbed by a dc voltage source $v(t) = V_{dc}$ that has a periodic current $i(t)$ is $P_{dc}$

$$P_{dc} = \frac{1}{T} \int_{t_0}^{t_0+T} v(t)i(t) \, dt = \frac{1}{T} \int_{t_0}^{t_0+T} V_{dc} \, i(t) \, dt$$

$$P_{dc} = V_{dc} \left[ \frac{1}{T} \int_{t_0}^{t_0+T} i(t) \, dt \right]$$

• Average power absorbed by a dc voltage source is the product of the voltage and the average current

$$P_{dc} = V_{dc}I_{avg}$$

• Average power absorbed by a dc source $i(t) = I_{dc}$

$$P_{dc} = V_{avg}I_{dc}$$
Inductors

- For periodic currents and voltages,
  \[
i(t + T) = i(t) \\
v(t + T) = v(t)
\]
- Voltage-current relationship for the inductor
  \[
i(t_o + T) = \frac{1}{L} \int_{t_o}^{(t_o+T)} V_L(t) \, dt + i(t_o)
\]
- Energy stored in an inductor
  \[
w(t) = \frac{1}{2} Li^2(t)
\]
- For periodic inductor current, stored energy at the end of one period is the same as at the beginning
- No net energy transfer indicates that the average power absorbed by an inductor is zero for steady-state periodic operation
  \[
P_L = 0
\]
  \[
i(t_o + T) - i(t_o) = \frac{1}{L} \int_{t_o}^{(t_o+T)} V_L(t) \, dt
\]
  \[
  avg[V_L(t)] = V_L = \frac{1}{T} \int_{t_o}^{(t_o+T)} V_L(t) \, dt = 0
\]

For periodic currents, the average voltage across an inductor is zero
Capacitors

- Voltage-current relationship for the inductor

\[ v(t_0 + T) = \frac{1}{C} \int_{t_0}^{(t_0+T)} i_c(t)dt + v(t_0) \]

- Energy stored in a capacitor

\[ w(t) = \frac{1}{2} C v^2(t) \]

- For a periodic capacitor voltage, the stored energy is the same at the end of a period as at the beginning.

- No net energy transfer indicates that the average power absorbed by the capacitor is zero for steady-state periodic operation

\[ P_c = 0 \]

\[ v(t_0 + T) - v(t_0) = \frac{1}{C} \int_{t_0}^{(t_0+T)} i_c(t)dt = 0 \]

\[ avg[i_c(t)] = I_c = \frac{1}{T} \int_{t_0}^{(t_0+T)} i_c(t)dt = 0 \]

For periodic voltages, the average current in a capacitor is zero
Inductors and Capacitors

Example
Determine the voltage, instantaneous power, and average power for the inductor

• Remember average inductor voltage is zero
• \( p(t) = v(t)i(t) \)

Average inductor power is 0
Energy recovery

- For $0 < t < t_1$

$$v_L = V_{CC}$$

$$i_s(t) = i_L(t)$$

$$i_L(t) = \frac{1}{L} \int_0^t v_L(\lambda) \, d\lambda + i_L(0) = \frac{1}{L} \int_0^t V_{CC} \, d\lambda + 0 = \frac{V_{CC}t}{L}$$
Energy recovery

For $t_1 < t < T$

• Initial condition for inductor current

$$i_L(t_1) = \frac{V_{CC} t_1}{L}$$

$$i_L(t) = i_L(t_1)e^{-(t-t_1)/\tau} = \left(\frac{V_{CC} t_1}{L}\right)e^{-(t-t_1)/\tau} \quad t_1 < t < T$$

• Average power supplied by dc source

$$P_S = V_S I_S = V_{CC} \left[ \frac{1}{T} \int_0^T i_s(t) \, dt \right] = V_{CC} \left[ \frac{1}{T} \int_0^{t_1} \frac{V_{CC} t}{L} \, dt + \frac{1}{T} \int_{t_1}^T 0 \, dt \right] = \frac{(V_{CC} t_1)^2}{2LT}$$
Effective values: RMS

- Effective value of a voltage or current is also known as the root-mean-square (rms) value
- Effective value of a periodic voltage waveform is based on the average power delivered to a resistor
- For a periodic voltage across a resistor, effective voltage is the voltage that is as effective as the dc voltage in supplying average power

\[ P = \frac{V_{dc}^2}{R}; \quad P = \frac{V_{eff}^2}{R} \]

- Average resistor power

\[
P = \frac{1}{T} \int_{0}^{T} p(t) \, dt = \frac{1}{T} \int_{0}^{T} v(t)i(t) \, dt = \frac{1}{T} \int_{0}^{T} \frac{v^2(t)}{R} \, dt = \frac{1}{R} \left[ \frac{1}{T} \int_{0}^{T} v^2(t) \, dt \right]
\]

\[
P = \frac{V_{eff}^2}{R} = \frac{1}{R} \left[ \frac{1}{T} \int_{0}^{T} v^2(t) \, dt \right]
\]

\[ V_{eff} = V_{rms} = \sqrt{\frac{1}{T} \int_{0}^{T} v^2(t) \, dt} \]

\[ I_{rms} = \sqrt{\frac{1}{T} \int_{0}^{T} i^2(t) \, dt} \]
Effective values: RMS

- Example: Find the rms of a periodic pulse waveform

\[ v(t) = \begin{cases} V_m & 0 < t < DT \\ 0 & DT < t < T \end{cases} \]

\[
V_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T v^2(t) \, dt} = \sqrt{\frac{1}{T} \left( \int_0^{DT} V_m^2 \, dt + \int_{DT}^T 0^2 \, dt \right)} = \sqrt{\frac{1}{T} (V_m^2 DT)}
\]

\[
V_{\text{rms}} = V_m \sqrt{D}
\]
Effective values: RMS

• If a periodic voltage is the sum of two periodic voltage waveforms \( v(t) = v_1(t) + v_2(t) \), the rms value of \( v(t) \)

\[
V_{\text{rms}} = \frac{1}{T} \int_0^T v_1^2(t) dt + \frac{1}{T} \int_0^T 2v_1v_2 dt + \frac{1}{T} \int_0^T v_2^2 dt
\]

• For orthogonal functions

\[
V_{\text{rms}} = \frac{1}{T} \int_0^T v_1^2(t) dt + \frac{1}{T} \int_0^T v_2^2(t) dt
\]

\[
V_{\text{rms}} = \sqrt{V_{1,\text{rms}}^2 + V_{2,\text{rms}}^2}
\]

\[
V_{\text{rms}} = \sqrt{V_{1,\text{rms}}^2 + V_{2,\text{rms}}^2 + V_{3,\text{rms}}^2 + \ldots} = \sqrt{\sum_{n=1}^{N} V_{n,\text{rms}}^2}
\]
Effective values: RMS

Examples

\[ i(t) = \begin{cases} \frac{2I_m t - I_m}{t_1} & 0 < t < t_1 \\ \frac{-2I_m t + I_m(T + t_1)}{T - t_1} & t_1 < t < T \end{cases} \]

\[ I_{\text{rms}} = \sqrt{I_{1,\text{rms}}^2 + I_{2,\text{rms}}^2} = \sqrt{\left(\frac{I_m}{\sqrt{3}}\right)^2 + I_{\text{dc}}^2} = \sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 + 3^2} = 3.22 \text{ A} \]

\[ I_{\text{rms}} = \frac{I_m}{\sqrt{3}} \]
Apparent power and power factor

- Apparent power is the product of rms voltage and rms current magnitudes
  \[ S = V_{rms} I_{rms} \]

- Power factor is the ratio of average power to apparent power
  \[ pf = \frac{P}{S} = \frac{P}{V_{rms} I_{rms}} \]

\[ pf = \cos \Phi \]

- Lagging power factor
  - Current lags behind voltage
  - Circuit is inductive

- Leading power factor
  - Current leads voltage
  - Circuit is capacitive

- Unity power factor
  - Current and voltage in phase
  - Ideal
Power computations for sinusoidal ac circuits

• For linear circuits that have sinusoidal sources, all steady-state voltages and currents are sinusoids

\[ v(t) = V_m \cos(\omega t + \theta) \]
\[ i(t) = I_m \cos(\omega t + \Phi) \]

• Instantaneous power

\[ p(t) = [V_m \cos(\omega t + \theta)][I_m \cos(\omega t + \Phi)] \]
\[ p(t) = \frac{I_m V_m}{2} [\cos(2\omega t + \theta + \Phi) + \cos(\theta - \Phi)] \]

• Average power

\[ P = \frac{1}{T} \int_0^T p(t)dt = \frac{I_m V_m}{2} \frac{1}{T} \int_0^T [\cos(2\omega t + \theta + \Phi) + \cos(\theta - \Phi)]dt \]

\[ P = \frac{I_m V_m}{2} \cos(\theta - \Phi) \]
\[ P = V_{rms} I_{rms} \cos(\theta - \Phi) \]

(\(\theta - \Phi\)) is the phase angle between voltage and current
Power computations for sinusoidal ac circuits

- Remember: In the steady state, no net power is absorbed by an inductor or a capacitor.
- Reactive power is commonly used in conjunction with voltages and currents for inductors and capacitors.
- Reactive power is characterized by energy storage during one-half of the cycle and energy retrieval during the other half.
  \[ Q = V_{rms}I_{rms} \sin(\theta - \Phi) \]
- Complex power combines real and reactive powers for ac circuits.
  \[ S = P + jQ = V_{rms}I_{rms}^* \]
- Apparent power in ac circuits is the magnitude of complex power.
  \[ S = |S| = \sqrt{P^2 + Q^2} \]
Power computations for nonsinusoidal periodic waveforms

- Power electronics circuits typically have voltages and/or currents that are periodic but not sinusoidal.
- Applying some special relationships for sinusoids to waveforms that are not sinusoids can cause errors.
- Option to use Fourier series to describe nonsinusoidal periodic waveforms in terms of a series of sinusoids.
- A periodic function \( f(t) \) can be expressed in trigonometric form.
- Sines and cosines can be combined into one sinusoid.

\[
\begin{align*}
    f(t) &= a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos (n\omega_0 t) + b_n \sin (n\omega_0 t) \right] \\
    f(t) &= a_0 + \sum_{n=1}^{\infty} C_n \cos (n\omega_0 t + \theta_n) \quad \text{or} \quad f(t) = a_0 + \sum_{n=1}^{\infty} C_n \sin (n\omega_0 t + \theta_n)
\end{align*}
\]

\[
C_n = \sqrt{a_n^2 + b_n^2} \quad \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \quad \theta_n = \tan^{-1}\left(\frac{d_n}{b_n}\right)
\]

- \text{rms value of } f(t)

\[
F_{\text{rms}} = \sqrt{\sum_{n=0}^{\infty} F_{n,\text{rms}}^2} = \sqrt{a_0^2 + \sum_{n=1}^{\infty} \left( \frac{C_n}{\sqrt{2}} \right)^2}
\]
Power computations for nonsinusoidal periodic waveforms

• Average power

\[ v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \theta_n) \]

\[ i(t) = I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t + \phi_n) \]

\[ P = \frac{1}{T} \int_0^T p(t) \, dt \]

• Average power for nonsinusoidal periodic voltage and current waveforms

\[ P = \sum_{n=0}^{\infty} P_n = V_0 I_0 + \sum_{n=1}^{\infty} V_{n,\text{rms}} I_{n,\text{rms}} \cos(\theta_n - \phi_n) \quad \text{or} \quad P = V_0 I_0 + \sum_{n=1}^{\infty} \left( \frac{V_{n,\text{max}} I_{n,\text{max}}}{2} \right) \cos(\theta_n - \phi_n) \]

Total average power is the sum of the powers at the frequencies in the Fourier series.
Power computations for nonsinusoidal periodic waveforms

• For a nonsinusoidal source applied to a linear load, use superposition to determine power absorbed by load

• Nonsinusoidal periodic voltage is equivalent to the series combination of the Fourier series voltages

\[ V_m \cos(n\omega_0 t + \theta_n) \]
\[ V_1 \cos(\omega_0 t + \theta_1) \]
\[ V_{dc} \]

\[ \rightarrow \]

Load

\[ i(t) \]

\[ = \]

\[ v(t) \]

load
Power computations for nonsinusoidal periodic waveforms

Example

Determine the power absorbed by the load

\[ v(t) = 10 + 20 \cos(2\pi60t - 25^\circ) + 30 \cos(4\pi60t + 20^\circ) \, V \]

- Current at source frequency should be computed separately

\[ I_o = \frac{V_o}{R} = \frac{10}{5} = 2 \, A \]

- Amplitudes of ac current terms computed from phasor analysis

\[ I_1 = \frac{V_1}{R + j\omega_1L} = \frac{20 \angle (-25^\circ)}{5 + j(2\pi60)(0.015)} = 2.65 \angle (-73.5^\circ) \, A \]

\[ I_2 = \frac{V_2}{R + j\omega_2L} = \frac{30 \angle 20^\circ}{5 + j(4\pi60)(0.015)} = 2.43 \angle (-46.2^\circ) \, A \]
Power computations for nonsinusoidal periodic waveforms

Example

- Load current
  \[ i(t) = 2 + 2.65 \cos(2\pi 60t - 73.5^\circ) + 2.43 \cos(4\pi 60t - 46.2^\circ) \text{ A} \]

- Power at each frequency in Fourier series
  \[ P_0 = (10 \text{ V})(2 \text{ A}) = 20 \text{ W} \]
  \[ \omega = 2\pi 60: \quad P_1 = \frac{(20)(2.65)}{2} \cos(-25^\circ + 73.5^\circ) = 17.4 \text{ W} \]
  \[ \omega = 4\pi 60: \quad P_2 = \frac{(30)(2.43)}{2} \cos(20^\circ + 46^\circ) = 14.8 \text{ W} \]

- Total power
  \[ P = 20 + 17.4 + 14.8 = 52.2 \text{ W} \]

- Power absorbed by load can be computed from \( I_{\text{rms}}^2 R \)
  \[ P = I_{\text{rms}}^2 R = \left[ 2^2 + \left( \frac{2.65}{\sqrt{2}} \right)^2 + \left( \frac{2.43}{\sqrt{2}} \right)^2 \right] 5 = 52.2 \text{ W} \]
Power computations for nonsinusoidal periodic waveforms

• If a sinusoidal voltage source is applied to a nonlinear load, current waveform can be represented as a Fourier series

\[ v(t) = V_1 \sin(\omega_0 t + \theta_1) \]

\[ i(t) = I_0 + \sum_{n=1}^{\infty} I_n \sin(n \omega_0 t + \phi_n) \]

\[ P = V_0 I_0 + \sum_{n=1}^{\infty} \left( \frac{V_{n, \text{max}} I_{n, \text{max}}}{2} \right) \cos(\theta_n - \phi_n) \]

\[ = (0)(I_0) + \left( \frac{V_1 I_1}{2} \right) \cos(\theta_1 - \phi_1) + \sum_{n=2}^{\infty} \left( \frac{V_{n, \text{max}} I_{n, \text{max}}}{2} \right) \cos(\theta_n - \phi_n) \]

\[ = \left( \frac{V_1 I_1}{2} \right) \cos(\theta_1 - \phi_1) = V_{1, \text{rms}} I_{1, \text{rms}} \cos(\theta_1 - \phi_1) \]

Power factor, \( pf = \frac{P}{S} = \frac{V_{\text{rms}} I_{\text{rms}}}{V_{1, \text{rms}} I_{1, \text{rms}}} \cos(\theta_1 - \phi_1) \)

\[ I_{\text{rms}} = \sqrt{\sum_{n=0}^{\infty} I_{n, \text{rms}}^2} = \sqrt{I_0^2 + \sum_{n=1}^{\infty} \left( \frac{I_n}{\sqrt{2}} \right)^2} \]

Only nonzero power term is at the frequency of the applied voltage
Power computations for nonsinusoidal periodic waveforms

- Distortion factor - ratio of the rms value of the fundamental frequency to the total rms value
- Distortion factor represents the reduction in power factor due to the nonsinusoidal property of the current
- Total harmonic distortion (THD) is another term used to quantify the nonsinusoidal property of a waveform
- THD is the ratio of the rms value of all the nonfundamental frequency terms to the rms value of the fundamental frequency term
- Distortion factor can be expressed as $DF = \frac{I_{1,\text{rms}}}{I_{\text{rms}}}$

- Power factor can also be expressed as $pf = [\cos (\theta_f - \phi_f)]DF$
- THD can be expressed as

$$THD = \sqrt{\frac{\sum_{n \neq 1} I_{n,\text{rms}}^2}{I_{1,\text{rms}}^2}} \quad \text{or} \quad THD = \sqrt{\frac{\sum_{n \neq 1} I_{n,\text{rms}}^2}{\sqrt{\sum_{n \neq 1} I_{n,\text{rms}}^2}}}$$

Distortion factor can also be expressed as

$$DF = \sqrt{\frac{1}{1 + (\text{THD})^2}}$$
Power computations for nonsinusoidal periodic waveforms

Example

• A sinusoidal voltage source of \( v(t) = 100 \cos(377t) \) V is applied to a nonlinear load, resulting in a nonsinusoidal current which is expressed in Fourier series form

\[
i(t) = 8 + 15 \cos(377t + 30^\circ) + 6 \cos[2(377)t + 45^\circ] + 2 \cos[3(377)t + 60^\circ]\]

Determine

• power absorbed by the load
• power factor of the load
• distortion factor of the load current
• total harmonic distortion of the load current
Power computations for nonsinusoidal periodic waveforms

Solution

• $v(t) = 100 \cos(377t)$ and $i(t) = 8 + 15 \cos (377t + 30^\circ) + 6 \cos [2(377)t + 45^\circ] + 2 \cos [3(377)t + 60^\circ]$

• Power absorbed by the load- power absorbed at each frequency in Fourier series

$$P = (0)(8) + \left( \frac{100}{\sqrt{2}} \right) \left( \frac{15}{\sqrt{2}} \right) \cos 30^\circ + (0) \left( \frac{6}{\sqrt{2}} \right) \cos 45^\circ + (0) \left( \frac{2}{\sqrt{2}} \right) \cos 60^\circ$$

$$P = \left( \frac{100}{\sqrt{2}} \right) \left( \frac{15}{\sqrt{2}} \right) \cos 30^\circ = 650 \text{ W}$$

• rms voltage

$$V_{\text{rms}} = \frac{100}{\sqrt{2}} = 70.7 \text{ V}$$

$$I_{\text{rms}} = \sqrt{8^2 + \left( \frac{15}{\sqrt{2}} \right)^2 + \left( \frac{6}{\sqrt{2}} \right)^2 + \left( \frac{2}{\sqrt{2}} \right)^2} = 14.0 \text{ A}$$

• Power factor

$$p_f = \frac{I_{1,\text{rms}} \cos(\theta_1 - \phi_1)}{I_{\text{rms}}} = \frac{\left( \frac{15}{\sqrt{2}} \right) \cos(0 - 30^\circ)}{14.0} = 0.66$$

or

$$\frac{P}{V_{\text{rms}}I_{\text{rms}}} = \frac{650}{70.7 \times 14.0} = 0.66$$
Power computations for nonsinusoidal periodic waveforms

Solution

• $v(t) = 100 \cos(377t)$ and $i(t) = 8 + 15 \cos(377t + 30^\circ) + 6 \cos[2(377)t + 45^\circ] + 2 \cos[3(377)t + 60^\circ]$

• Distortion factor can be expressed as

$$DF = \frac{I_{1,rms}}{I_{rms}} = \frac{15}{\frac{\sqrt{2}}{14.0}} = 0.76$$

• Total harmonic distortion of load current

$$THD = \sqrt{\frac{I_{2,rms}^2 - I_{1,rms}^2}{I_{1,rms}^2}} = \sqrt{\frac{14^2 - \left(\frac{15}{\sqrt{2}}\right)^2}{\left(\frac{15}{\sqrt{2}}\right)^2}} = 0.86 = 86\%$$
Summary

- **Instantaneous power** $p(t)$ is $p(t) = v(t) i(t)$

- **Average power** $P = \frac{1}{T} \int_{t_0}^{t_0+T} p(t) \, dt$

- For inductors and capacitors that have periodic voltages and currents, the average power is zero. Instantaneous power is generally not zero because the device store energy and then returns energy to the circuit.

- For periodic currents, the average voltage across an inductor is zero.

- For periodic voltages, the average current in a capacitor is zero.

- For nonsinusoidal periodic waveforms, average power may be computed from the basic definition, or the Fourier series method may be used. The Fourier series method treats each frequency in the series separately and uses superposition to compute total power.

- Power factor is the ratio of average power to apparent power

- Apparent power is the product of rms voltage and current

- The rms value is the root-mean-square or effective value of a voltage or current waveform