State Variable Filters and Mason Rule

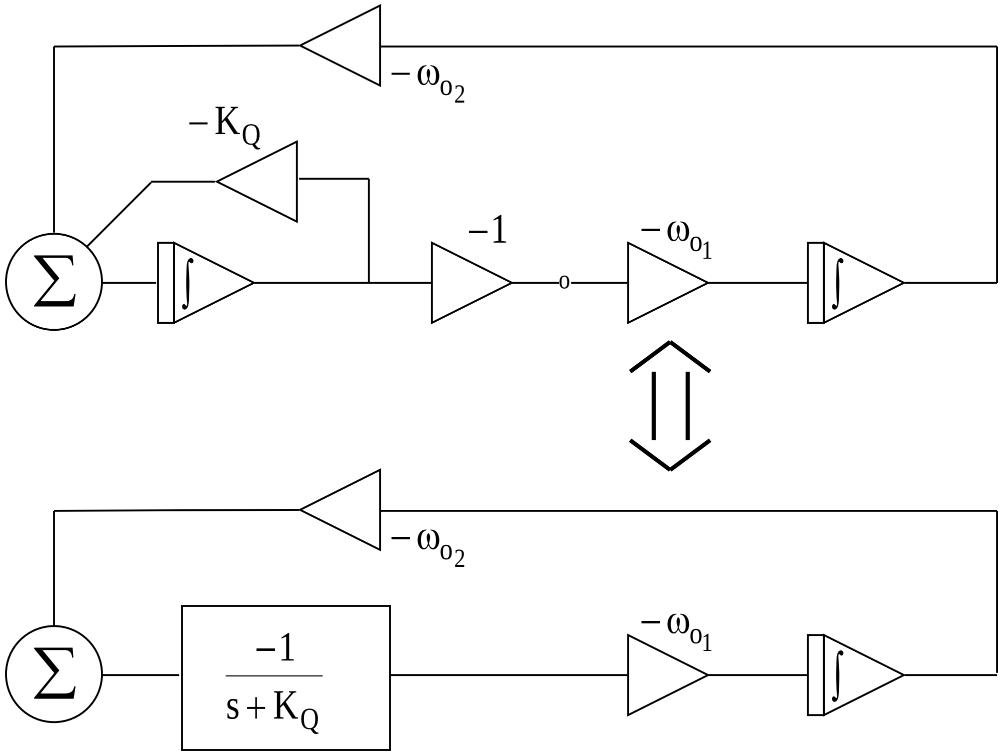
- A second order filter is also known as a Biquad.
- A second – order filter consists of a two integrator loop of one lossless and one lossy integrator
- Using ideal components all the biquad topologies have the same transfer function.
- Biquad with real components are topology dependent .

➢ We will cover the following material:
  - Biquad topologies
  - Voltage scaling to equalize signal at all filter nodes.
  - State variable Biquad derivation.
  - Tow-Thomas Biquad Design Procedure
  - Fully Balanced topologies using single ended Op Amps
  - Non-ideal integrators and Biquads.
  - **Mason Rule to obtain transfer functions by inspection**

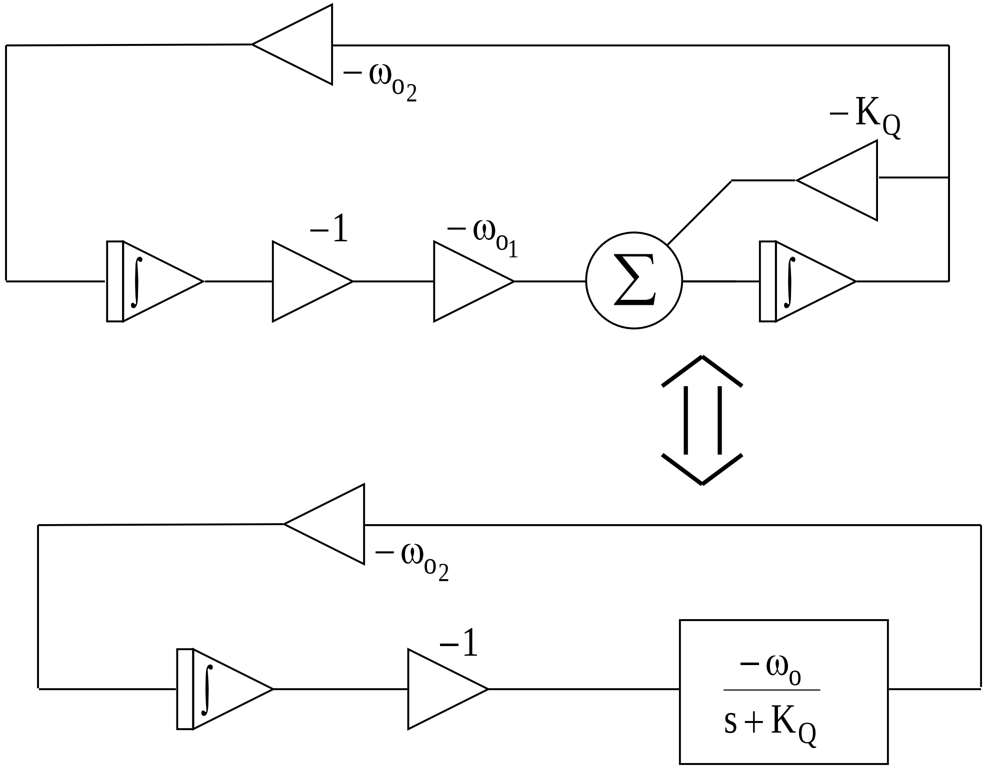
TWO-INTEGRATOR LOOP FAMILIES

Two-integrator loops consists of one lossless and one
lossy integrator, furthermore one is positive and the other is negative.

Why do we consider different filter topologies to obtain
the same (ideal) transfer function ?



Family 1a



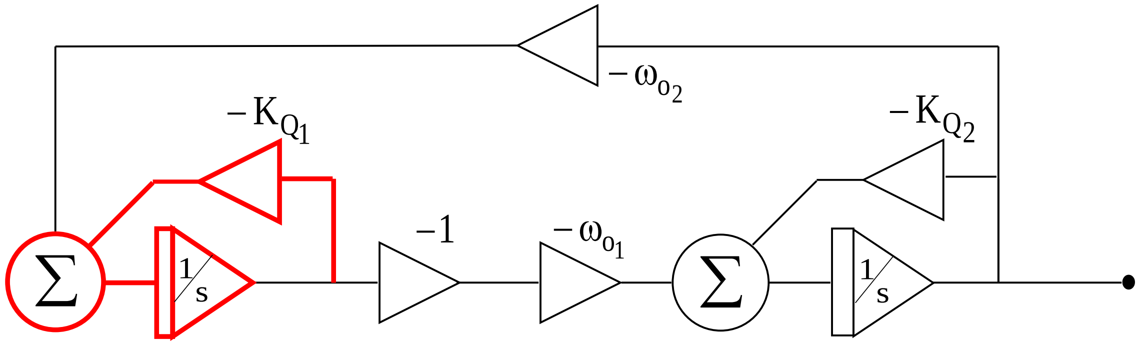
Family 1b

$$D(s) = 1 + \frac{K_Q}{s} + \frac{\omega_{o_1}\omega_{o_2}}{s^2}$$

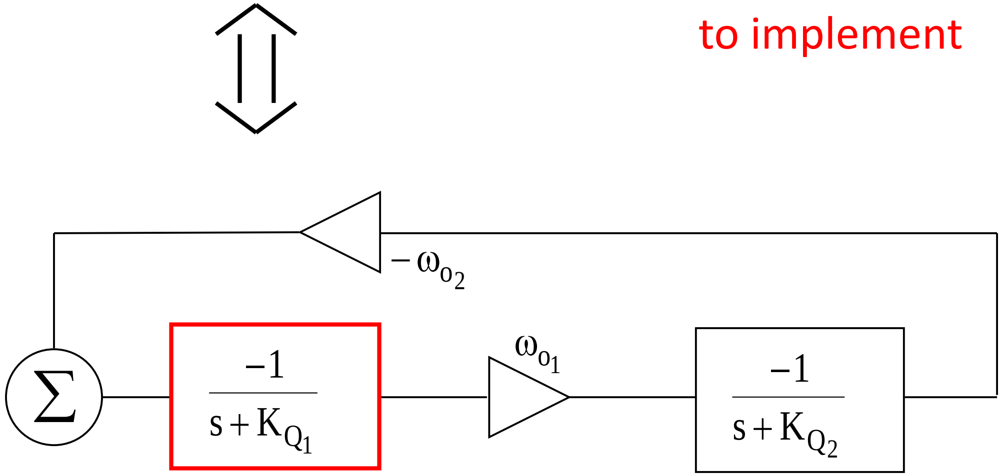
$$s^2 D(s) = s^2 + K_Q s + \omega_{o_1}\omega_{o_2}; \quad S^{BW}_{K_Q} = 1 \cdot$$

All loops must be negative. $S^{\omega_0^2}_{\omega_{o_1}} = 1$

Topology using two lossy integrators



- Lossy integrators are easy to implement



Two-Integrator Loop Family 2, ideal to avoid CMF in Fully Differential Structures

$$D(s) = 1 + \frac{K_{Q_1}}{s} + \frac{K_{Q_2}}{s} + \frac{\omega_{o_1}\omega_{o_2}}{s^2} + \frac{K_{Q_1}K_{Q_2}}{s^2}$$

$$s^2 D(s) = s^2 + (K_{Q_1} + K_{Q_2})s + (\omega_{o_1}\omega_{o_2} + K_{Q_1}K_{Q_2})$$

$$\omega_o^2 = \omega_{o_1}\omega_{o_2} + K_{Q_1}K_{Q_2} \qquad BW = \frac{\omega_o}{Q} = K_{Q_1} + K_{Q_2}$$

Low sensitivity structure:

$$S_{K_{Q_1}}^{BW} = 1 \cdot \frac{K_{Q_1}}{BW}; \quad S_{K_{Q_1}}^{BW} = \frac{K_{Q_1}}{K_{Q_1} + K_{Q_2}} = \frac{1}{1 + \frac{K_{Q_2}}{K_{Q_1}}}$$

$$S_{\omega_{o_1}}^{\omega_o^2} = \omega_{o_2} \cdot \frac{\omega_{o_1}}{\omega_{o_1} + \omega_{o_2} + K_{Q_1}K_{Q_2}} = \frac{1}{1 + \frac{K_{Q_1}K_{Q_2}}{\omega_{o_1}\omega_{o_2}}}$$

Self-Loop two integrator loop

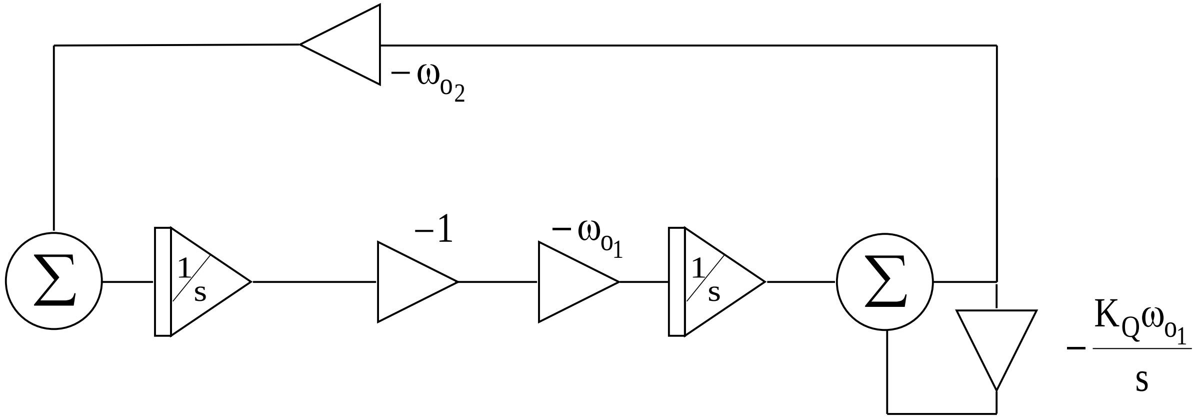
Design Equations:

Given ω_o, BW, PICK K_{Q_2} and ω_{o_2}

$$K_{Q_1} = BW - K_{Q_2}$$

$$\omega_o^2 = \omega_{o_1}\omega_{o_2} + (BW - K_{Q_2})K_{Q_2}$$

$$\omega_{o_1}\omega_{o_2} = \omega_o^2 + (K_{Q_2} - BW)K_{Q_2}$$



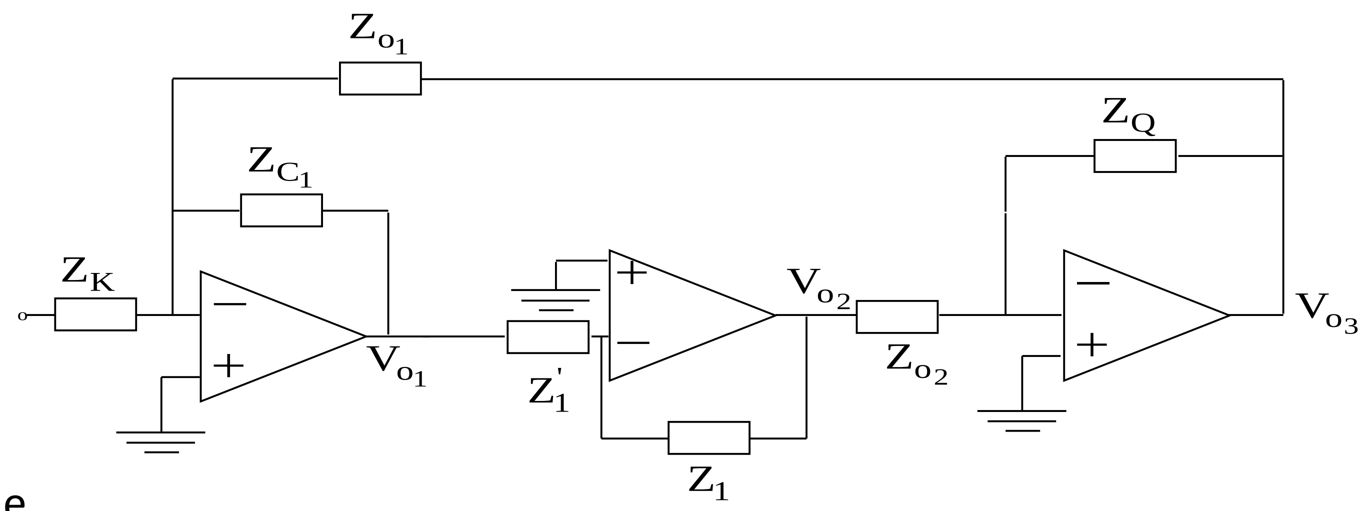
Family 3 Self-Loop

$$D(s) = 1 + \frac{\omega_{o_1}\omega_{o_2}}{s^2} + \frac{K_Q\omega_{o_1}}{s}$$

$$s^2D(s) = s^2 + K_Q\omega_{o_1}s + \omega_{o_1}\omega_{o_2}$$

A self loop can be implemented by adding a resistor to one (lossless) integrator.

SCALING for Active-RC, MOSFET-C and SC implementations



An example.

$$V_{o_1} \rightarrow k\, V_{o_1}$$

Modify all the impedances connected to the output under consideration.

$$Z_{C_1} \rightarrow k\ Z_{C_1} \quad \Rightarrow \frac{k}{SC_1} = \frac{1}{SC_1/k} \Rightarrow C_1 \rightarrow C_1/k$$

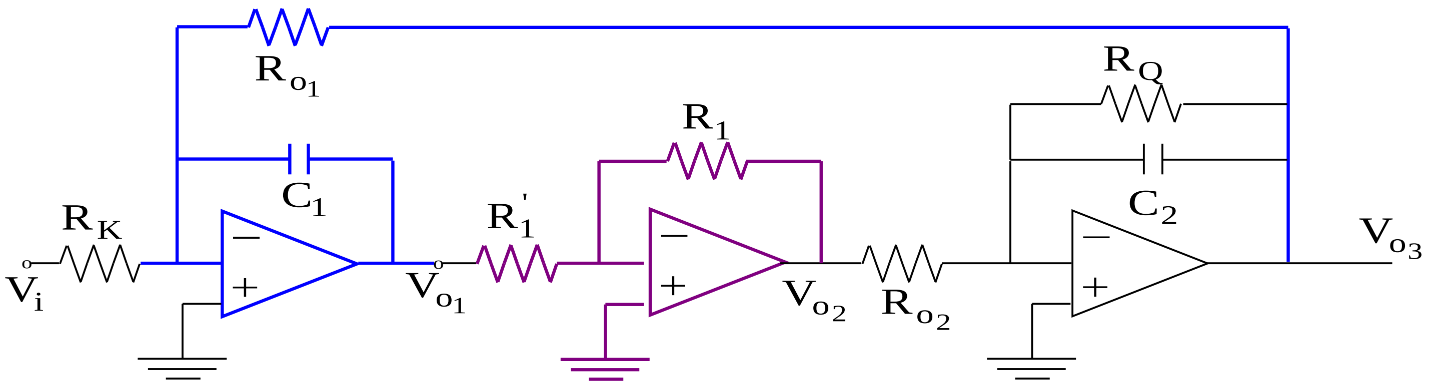
$$Z_1' \rightarrow k\ Z_1' \quad \Rightarrow k\ R_1' \Rightarrow R_1' \rightarrow k\ R_1'$$

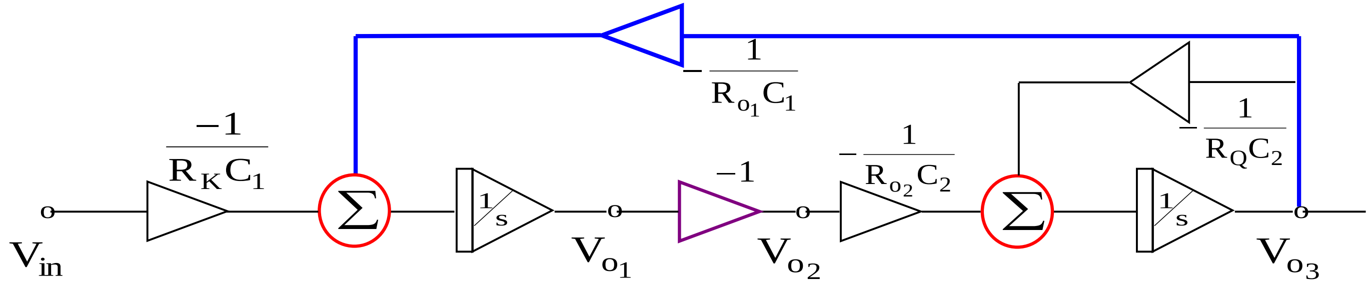
Note that the voltages V_{02} and V_{03} were not modified

VOLTAGE SWING SCALING

A good filter design approach requires to have a proper voltage swing at a frequency range for all internal and output nodes of the filter.

Let us consider the different types of filter implementations.





$$s^2D(s) = s^2 + s\frac{1}{R_QC_2} + \frac{1}{R_{o_1}R_{o_2}C_1C_2}\cdot\frac{R_1}{R_1'}$$

$$H_1(s) = \frac{V_{o_1}}{V_{in}} = \frac{-\frac{1}{R_KC_1s}\left(1+\frac{1}{R_QC_2s}\right)s^2}{D(s)} = \frac{-\frac{1}{R_kC_1}\left(s+\frac{1}{R_QC_2}\right)}{s^2+s\frac{1}{R_QC_2}+\frac{1}{R_{o_1}R_{o_2}C_1C_2}}$$

$$H_2(s) = \frac{V_{o_2}}{V_{in}} = -H_1(s)$$

$$H_3(s) = \frac{V_{o_3}}{V_{in}} = \frac{\dfrac{-1}{R_K C_1 R_{o_2} C_2}}{S^2 + \dfrac{S}{R_Q C_2} + \dfrac{1}{R_{o_1} R_{o_2} C_1 C_2}}$$

Let us consider that we want

$$\left|H_3(j\omega_o)\right| = K$$

$$\left|H_3(j\omega_o)\right| = \frac{\dfrac{1}{R_K C_1 R_{o_2} C_2}}{\dfrac{\omega_o}{R_Q C_2}} = \frac{R_Q}{R_K C_1 R_{o_2} \omega_o} = K = \frac{Q}{\omega_o^2 R_K C_1 R_{o_2} C_2}$$

$$K \left| \begin{aligned} &= \frac{Q}{\omega_0 R_K C_1} \\ &\omega_o = \frac{1}{R_{o_1} C_1} = \frac{1}{R_o C_2} \end{aligned} \right.$$

Thus $\quad R_K = \dfrac{R_Q}{K C_1 R_{o_2} \omega_o} \quad ; \quad \omega_o^2 = \dfrac{1}{R_{o_1} R_{o_2} C_1 C_2} \cdot \dfrac{R_1}{R_1^{'}}$

$$H_1(j\omega_o) = \frac{-\frac{1}{R_K C_1}\left(j\omega_o + \frac{1}{R_Q C_2}\right)}{j\frac{\omega_o}{R_Q C_2}} = \frac{-\frac{R_Q C_2}{R_K C_1}\left(j\omega_o + \frac{1}{R_Q C_2}\right)}{j\omega_o}$$

$$\left|H_1(j\omega_o)\right| = \frac{R_Q C_2}{R_K C_1 \omega_o}\sqrt{\omega_0^2 + \left(\frac{1}{R_Q C_2}\right)^2} = \frac{R_Q}{R_K C_1}\sqrt{1 + \frac{1}{Q^2}}$$

NUMERICAL EXAMPLE

$$\omega_o = 1 \qquad Q = \frac{3}{4} \qquad K = 2$$

THEN

$$\frac{1}{R_Q C_2} = \frac{\omega_o}{Q} = \frac{1}{3/4} \quad ; \quad \frac{1}{R_{o_1} C_1} = \frac{1}{R_o C_2} = \omega_o = 1$$

$$R_K = \frac{5}{2} = 2.5$$

$$C_1 = C_2 = 1$$

$$R_{o_1} = R_{o_2} = 1$$

$$R_Q = 5$$

$$\left|H_1(j\omega_o)\right| = \frac{5\times 1}{2.5\times 1}\sqrt{1+\frac{1}{25}} \cong 10$$

TO VERIFY

$$\left|H_3(j\omega_o)\right| = \frac{2}{1/5} = 10$$

We want to make

$$|H_i(j\omega_o)| = |H_3(j\omega_o)|$$

$$V_{o_1} \rightarrow \frac{1}{5} V_{o_1}$$

$$\frac{1}{SC_1} \ \& \ R_1^1 \rightarrow \frac{1}{SC_1 5} \ \& \ \frac{R_1}{5} \Rightarrow \begin{array}{l} C_1 \rightarrow 5C_1 \\ R_1^{'} \rightarrow \dfrac{R_1^{'}}{5} \end{array}$$

Before scaling

$$V_{o_1} = V_i(-\frac{1}{SR_K C_1}) - \frac{V_{o_3}}{SR_{o_1} C_1} \quad ; \quad V_{o_2} = -V_{o_1}$$

After

$$V_{o_1} = V_i(-\frac{1}{SR_K C_1})\frac{1}{5} - (\frac{V_{o_3}}{SR_{o_1} C_1})\frac{1}{5} \quad ; \quad V_{o_2} = -V_{o_1}.$$

Thus we have modified V_{o_1} and V_{o_2} without changing V_{o_3}

This usually can be done when enough degrees of freedom exist.

STATE-VARIABLE FILTER ARCHITECTURES

- Derived from state-variable techniques to solve differential equations. The objective is to render an expression that can be implemented using integrator building blocks
- Example

$$H(s)=\frac{V_o(s)}{V_i(s)}=\frac{-Ks}{s^2+\frac{\omega_o}{Q}s+\omega_0^2} \qquad (1)$$

Where ω_o is the cut-off frequency, (ω_o/Q) is the bandwidth, Q is the quality (selectivity) factor.

Let us rewrite the above equation by multiplying

$$\frac{V_o(s)}{V_i(s)}=\frac{-\frac{K}{s}X(s)}{\left[1+\frac{\omega_o}{Q}\frac{1}{s}+\frac{\omega_o^2}{s}\right]X(s)} \qquad (2)$$

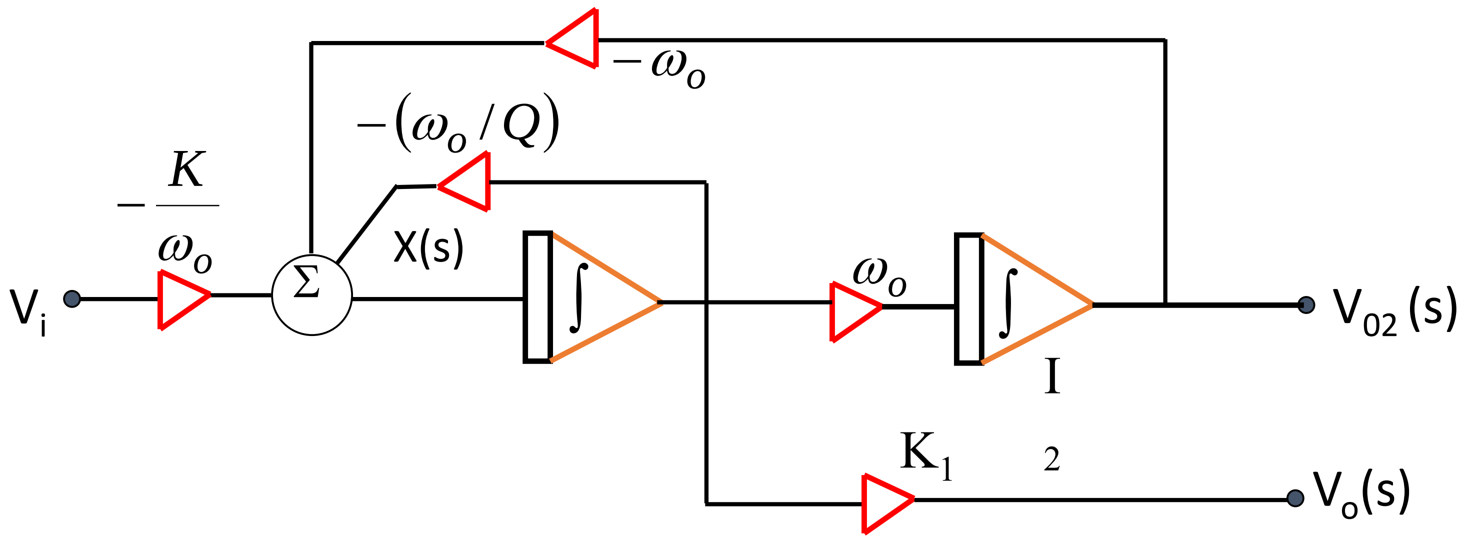
Both numerator and denominator by $X(s)/s^2$. Thus (2) can be split into

$$X(s) = V_i(s) - \frac{\omega_o}{Q}\frac{X(s)}{s} - \omega_o^2 \frac{X(s)}{s^2} \qquad (3a)$$

$$V_o(s) = -K\frac{X(s)}{s} \qquad (3b)$$

Observe that 1/s is an integral operator. Thus we implement (3) by using integral building block.

Remember that each integrator has its time-constant . We can associate more than one time-constant with each integrator.



Also observe that V_{02} correspond to a low-pass, V_o to a bandpass and X(s) to a high pass. K_1 could be 1 or any other suitable value

This two-integrator biquad consists of

– One lossless integrator (I_2)

– One Lossy integrator (I_1)

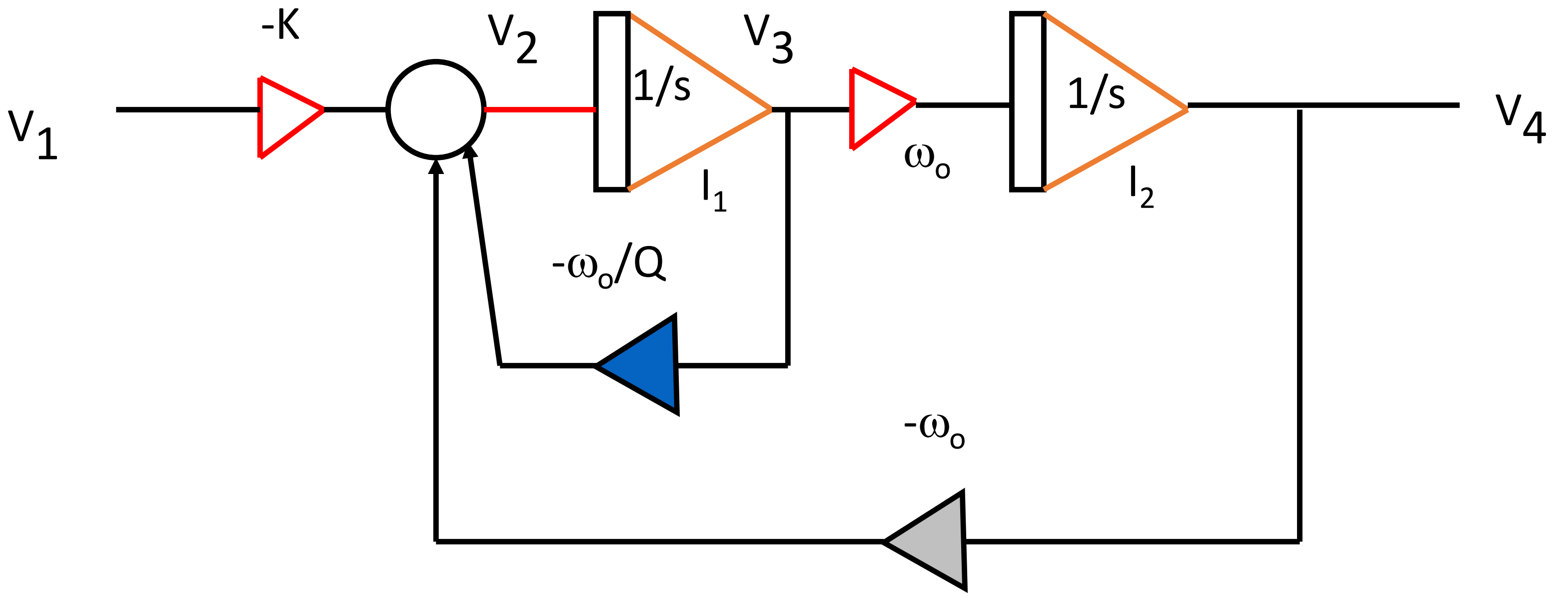
– Two negative closed loops: One determines the center frequency, the other the bandwidth (or Q).

– One of two integrators must be positive, the other negative.

–The lossy integrator must have a negative feedback.

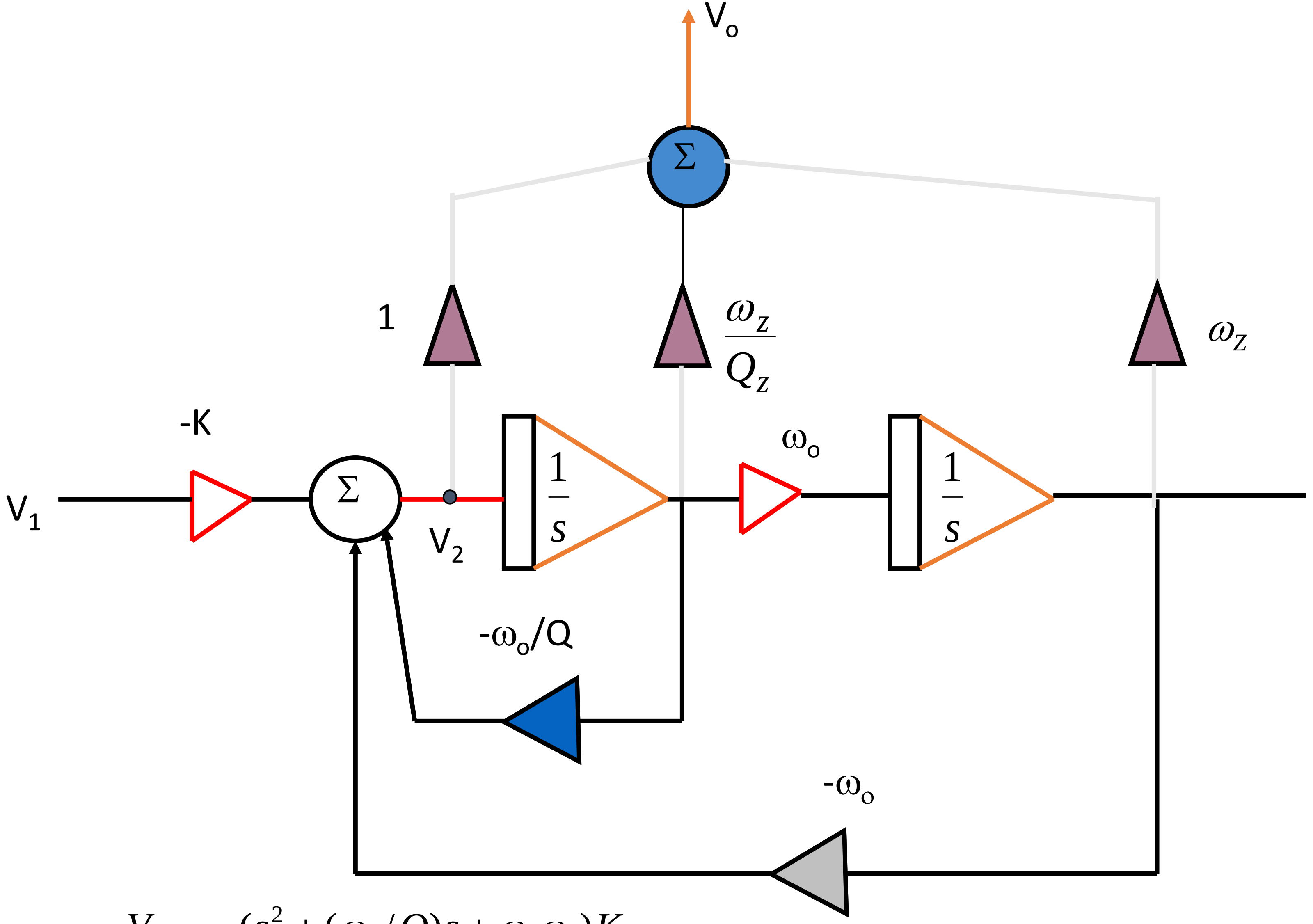
Second-Order Filter Structures

State-Variable Structure



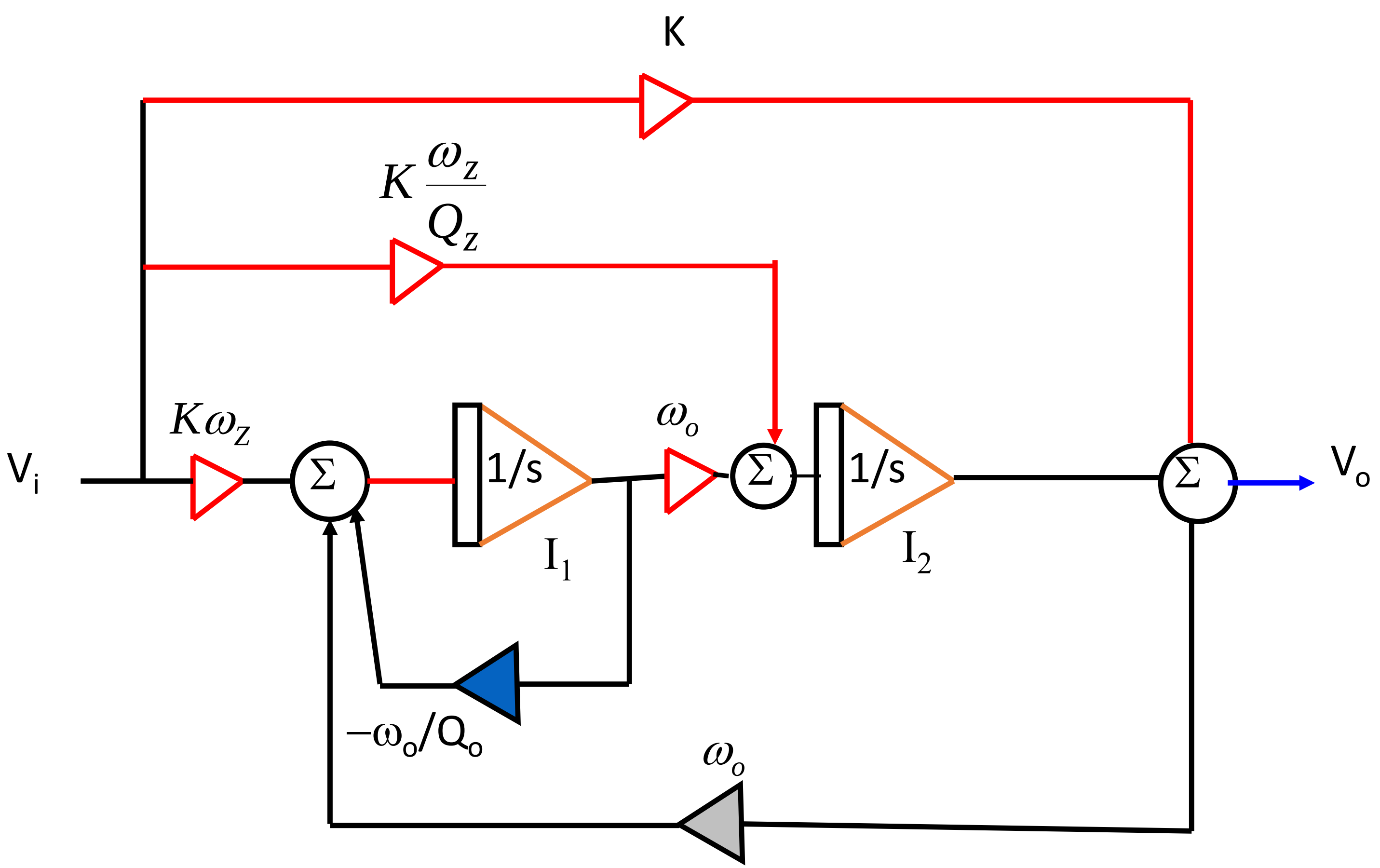
$$H(s) = \frac{V_4}{V_1} = \frac{-K\omega_o}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2}$$

Zero implementation by addition of outputs technique



$$H(s) = \frac{V_o}{V_1} = \frac{-(s^2 + (\omega_z / Q)s + \omega_z \omega_o)K}{s^2 + (\omega_o / Q)s + \omega_o^2}$$

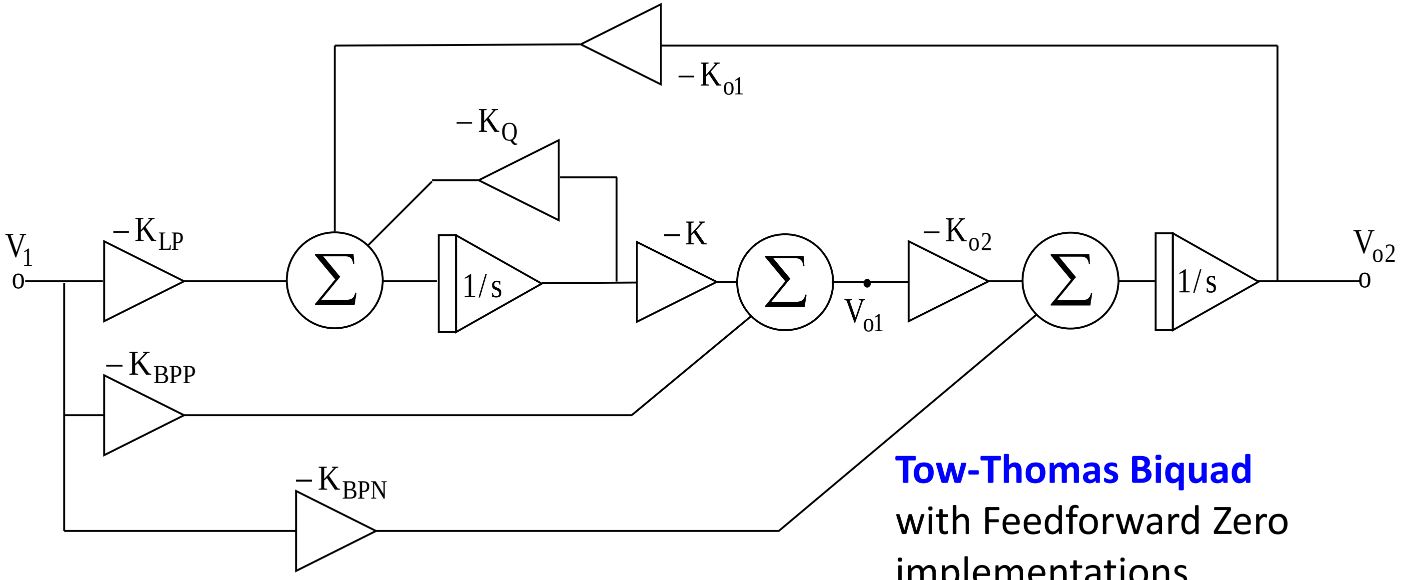
Feed-Forward Zeros Implementation



$$H(s) = \frac{V_o}{V_1} = K\frac{s^2 + (\omega_z / Q_z)s + \omega_z \omega_0}{s^2 + (\omega_o / Q_o)s + \omega_o^2}$$

Why H(s) is not correct?

Tow-Thomas Biquad



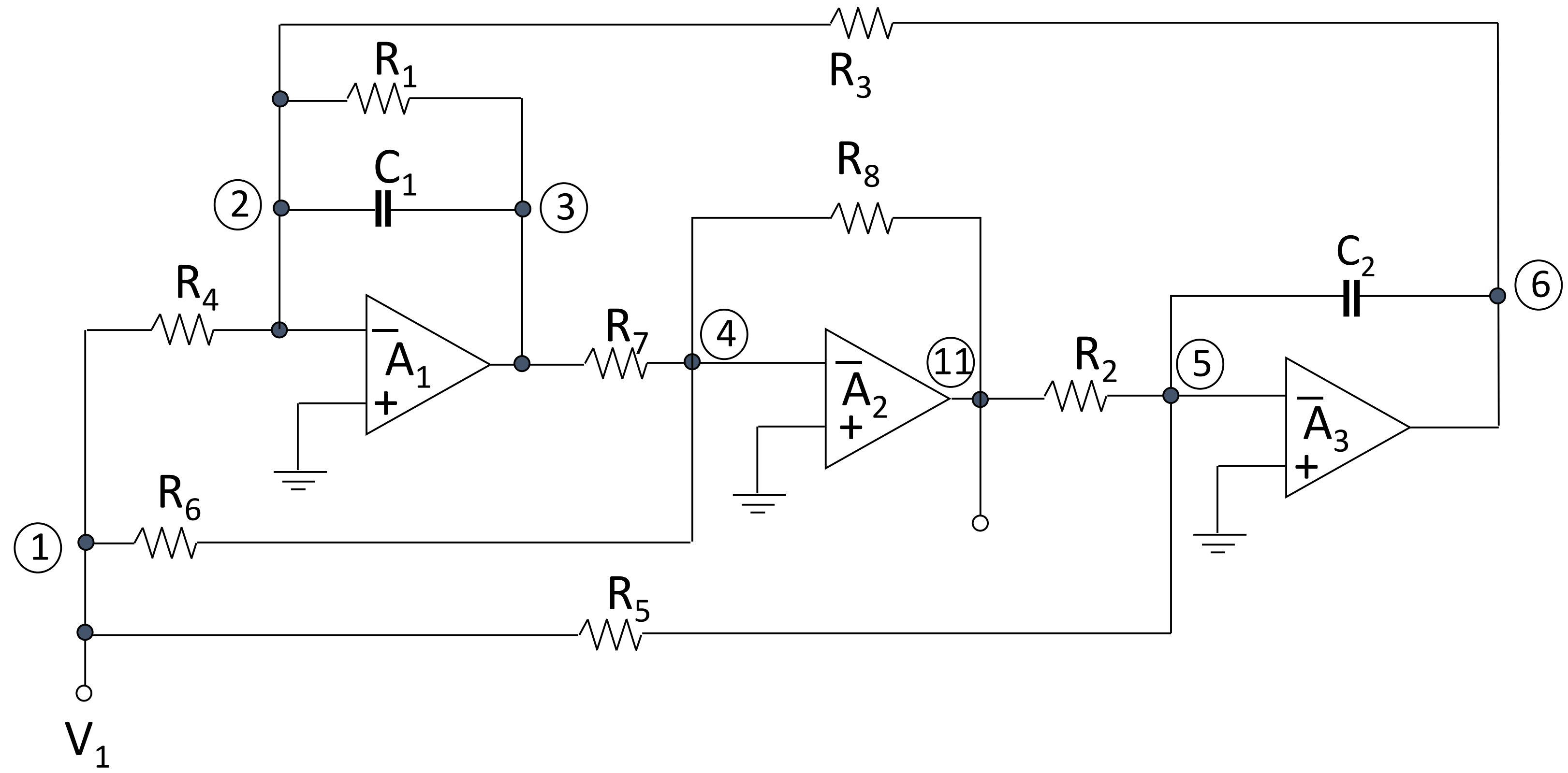
Tow-Thomas Biquad
with Feedforward Zero implementations

$$H_2(s)=\frac{V_{o_R}(s)}{V_1(s)}=\frac{-K_{LP}\;K\;K_{o2}/s^2+(K_{o2}\;K_{BPP}/s-K_{BPN}/s)(1+K_Q/s)}{1+\dfrac{K_Q}{s}+\dfrac{K\;K_{o1}\;K_{o2}}{s^2}}$$

$$H_2(s)=\frac{(s+K_Q)(K_{o2}K_{BPP}-K_{BPN})-K_{LP}K_{o2}}{s^2+K_Qs+KK_{o1}K_{o2}}$$

$$H_1(s)=\frac{V_{o1}(s)}{V_1(s)}=\frac{s^2+(K_Q-K_{LP}K/K_{BPP})s+K_{BPN}K_{o1}K/K_{BPP}}{s^2+K_Qs+KK_{o1}K_{o2}}(-K_{BPP})$$

Feedforward Tow-Thomas (TT) Biquad Circuit



Observe that the regular TT Biquad does not implement a highpass output

Description of the Parameters for the Tow-Thomas Filter

General Transfer Function

$$T(s) = -\frac{R_8}{R_6} \frac{s^2 + \left(\frac{1}{R_1 C_9} - \frac{1}{R_4 C_9} \frac{R_6}{R_7}\right) + \frac{R_6}{R_7} \frac{1}{R_3 R_5 C_9 C_{10}}}{s^2 + s\left(\frac{1}{R_1 C_9}\right) + \frac{R_8}{R_7} \frac{1}{R_3 R_2 C_9 C_{10}}}$$

where

$$\omega_p^2 = \frac{R_8}{R_7 R_2 R_3 C_9 C_{10}}, \qquad \omega_z^2 = \frac{R_6}{R_3 R_5 R_7 C_9 C_{10}}$$

$$Q_p = R_1 \sqrt{\frac{R_8 C_9}{R_2 R_3 R_7 C_{10}}}, \qquad Q_z = \sqrt{\frac{R_6 C_9}{R_3 R_5 R_7 C_{10}}} \Bigg/ \left(\frac{1}{R_1} - \frac{R_6}{R_4 R_7}\right)$$

and

$$|H_{HP}| = \frac{R_8}{R_6}, \qquad \text{for} \qquad R_1 = \frac{R_4 R_7}{R_6}, \quad R_5 \to \infty$$

$$|H_{BP}| = \frac{R_1 R_8}{R_4 R_7}, \qquad \text{for} \qquad R_5,\ R_6 \to \infty$$

$$|H_{LP}| = \frac{R_2}{R_5}, \qquad \text{for} \qquad R_4, R_6 \to \infty$$

For the bandstop (notch)

$$|H_{notch}| = \frac{R_8}{R_6}, \qquad \text{for} \qquad R_1 = \frac{R_4 R_7}{R_6}, \quad R_5 = \frac{R_6 R_2}{R_8}$$

Design Equations for the Tow-Thomas Filter

Let

$$R_3 = R$$

$$R_2 = a^2 R_3$$

$$R_7 = R_8 = R'$$

$$C_1 = C_2 = C$$

$$\omega_p = \frac{1}{aRC}, \quad \omega = \frac{1}{C}\sqrt{\frac{R_6}{R_5 R R'}}$$

$$Q_p = \frac{R_1}{aR}, \quad Q_z = \sqrt{\frac{R_6}{R_5 R R'}} \Bigg/ \left(\frac{1}{R_1} - \frac{R_6}{R_4 R'}\right)$$

$$|H_{HP}| = \frac{R'}{R_6}, \qquad \text{for} \qquad R_1 = \frac{R_4 R'}{R_6} = R_4 |H_{HP}|, \quad R_5 \to \infty$$

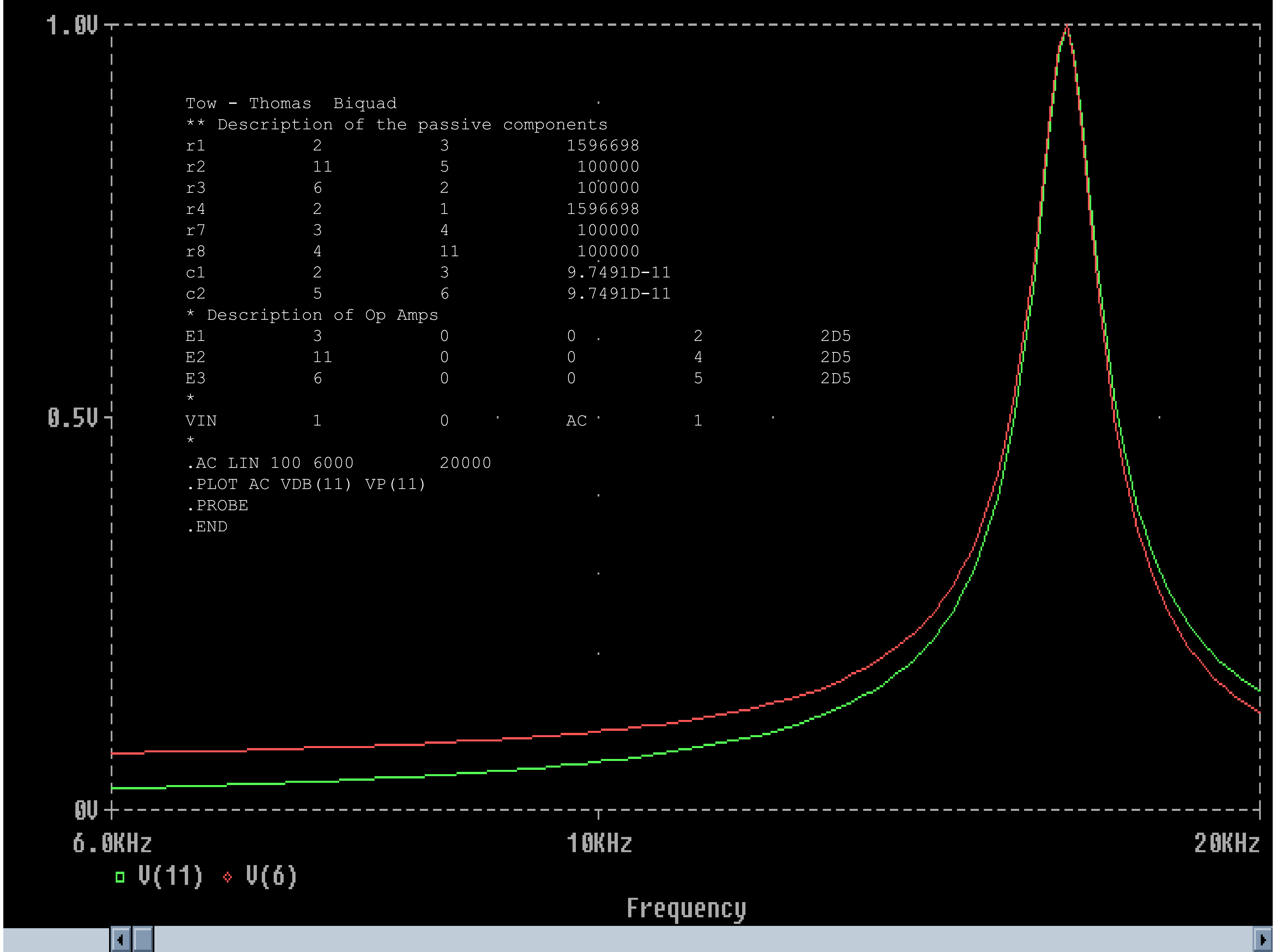
$$|H_{HP}| = \frac{R_1}{R_4}, \qquad \text{for} \qquad R_5, R_6 \to \infty$$

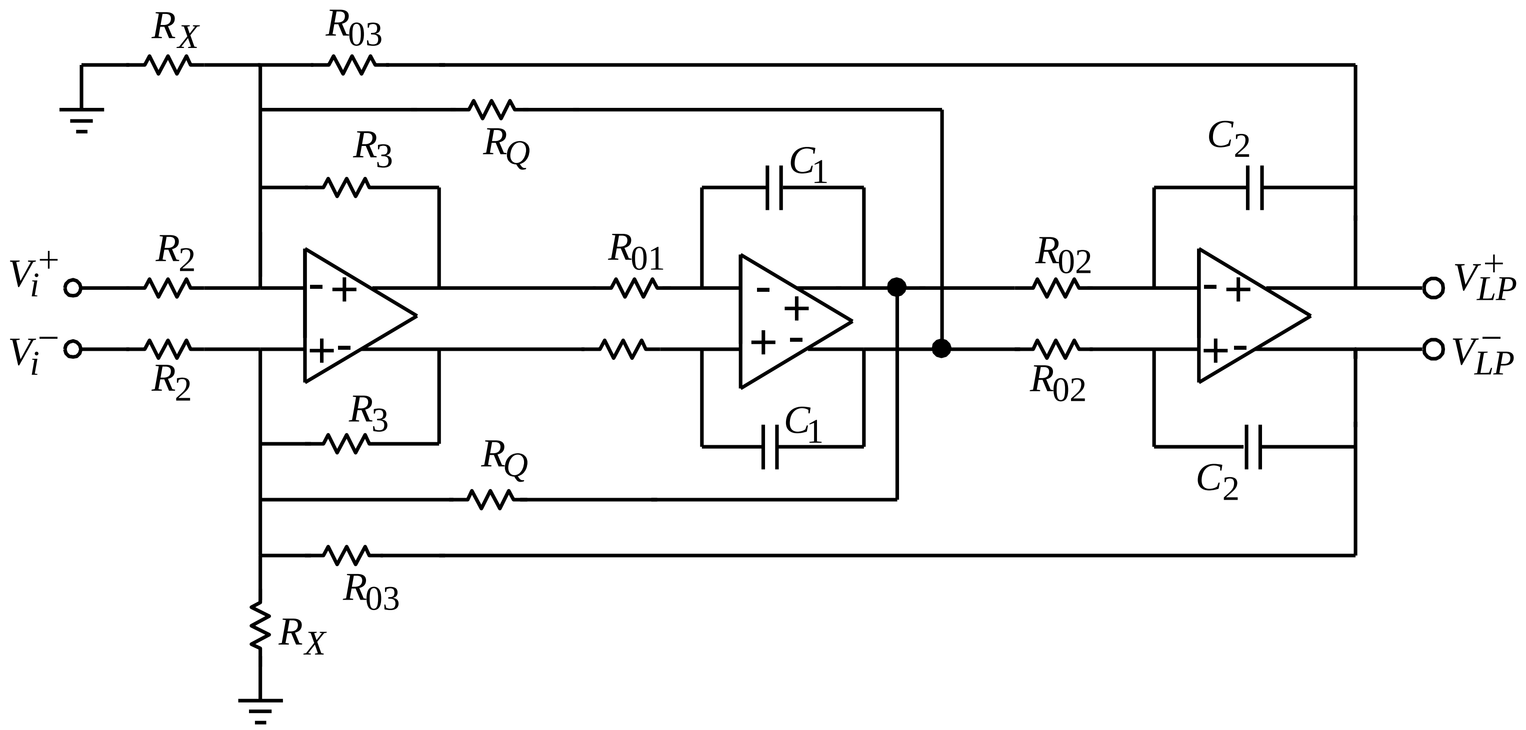
$$|H_{LP}| = \frac{a^2 R}{R_5}, \qquad \text{for} \qquad R_4, R_6 \to \infty$$

$$|H_{notch}| = \frac{R'}{R_6}, \qquad \text{for} \qquad R_1 = \frac{R_4 R'}{R_6}, \quad R_5 = \frac{a^2 R R_6}{R'}$$

PSPICE Input file of Tow Thomas Filter

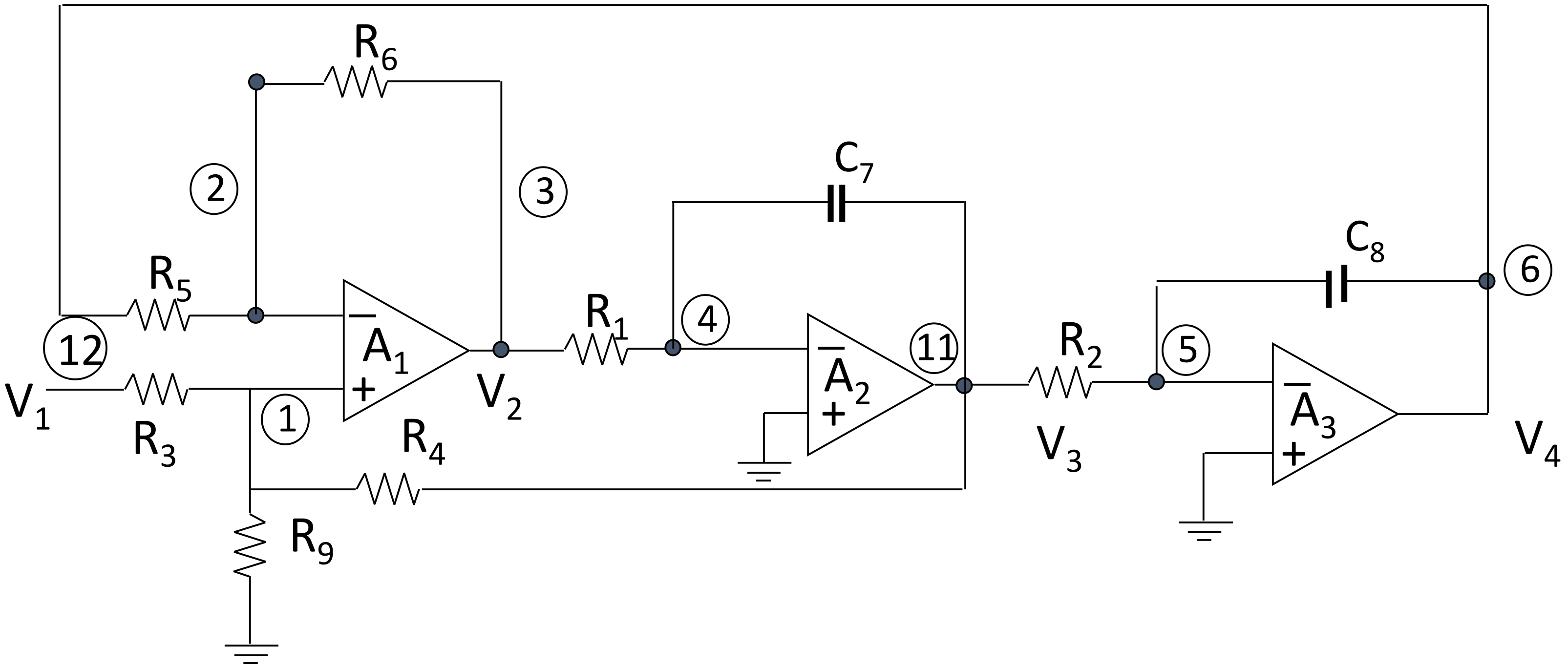
```
Tow - Thomas  Biquad
** Description of the passive components
r1      2       3       1596698
r2      11      5        100000
r3      6       2        100000
r4      2       1       1596698
r7      3       4        100000
r8      4       11       100000
c1      2       3       9.7491D-11
c2      5       6       9.7491D-11
* Description of Op Amps
E1      3       0       0       2       2D5
E2      11      0       0       4       2D5
E3      6       0       0       5       2D5
*
VIN     1       0       AC      1
*
.AC LIN 100 6000        20000
.PLOT AC VDB(11) VP(11)
.PROBE
.END
```


1.0V
0.5V
0V
Tow - Thomas Biquad
** Description of the passive components
r1 2 3 1596698
r2 11 5 100000
r3 6 2 100000
r4 2 1 1596698
r7 3 4 100000
r8 4 11 100000
c1 2 3 9.7491D-11
c2 5 6 9.7491D-11
* Description of Op Amps
E1 3 0 0 2 2D5
E2 11 0 0 4 2D5
E3 6 0 0 5 2D5
*
VIN 1 0 AC 1
*
.AC LIN 100 6000 20000
.PLOT AC VDB(11) VP(11)
.PROBE
.END
6.0KHz
10KHz
20KHz
V(11)
V(6)
Frequency



KHN Fully-Differential Version

Kerwin-Huelsman-Newcomb Biquad Circuit



Simple Design Equations for the KHN Filter

Let

$$R_4 = R_5 = b^2 R_6 = R$$

$$R_1 = R_2 = R'$$

$$C_7 = C_8 = C$$

$$\omega_p = \frac{1}{bR'C}$$

$$Q_p \frac{1 + R/R_3 + R/R_9}{\left(1 + 1/b^2\right) b}$$

$$|H_{HP}| = \frac{1 + 1/b^2}{1 + R_3/R + R_3/R_9}$$

$$|H_{BP}| = \frac{R}{R_3},$$

$$|H_{LP}| = \frac{R}{R_5,}$$

KHN Biquad Design Procedure

A simple design procedure is described as follows:

1. Assume that w_p, Q_p, and H=K are the design specifications.

2. Select convenient values* or R', C, and R to determine the values of R_1, R_2, C_7, C_8, R_4, and R_6.

3. Calculate the following element values:

$$b = \frac{1}{R'C\omega_p}$$

For the HP case:

$$R_3 = \frac{R}{bQ_p H_{HP}}$$

$$R_9 = \frac{R}{bQ_p\left[1+\left(1/b^2\right)-H_{HP}\right]-1}$$

* A rule of thumb for choosing R' and R is to make them proportional to $10f_p$, when $Q_p > 10$, or else make R' and R proportional to f_p. Then C should be made proportional to $1/R'\,\omega_p$.

For the BP case:

$$R_3 = \frac{R}{H_{BP}}$$

$$R_9 = \frac{R}{bQ_p\left[1+\left(1/b^2\right)-1-H_{BP}\right]}$$

For the LP case:

$$R_3 = \frac{bR}{Q_p H_{LP}}$$

$$R_9 = \frac{bR}{\left(1+b^2\right)Q_p - b - Q_p H_{LP}}$$

Example Design a bandpass modified KHN filter having a gain of H_{BP} =3, Q_P = 20, and $\omega_o = 2\pi \times 10^3$ r/s.

Procedure

1. Let us choose R=10 kΩ, C=0/01 μF, and R'=11.254 kΩ; that is $R_1 = R_2$ = 11.254kΩ and $R_4 = R_5$ = 10kΩ.

2. Since the values of R_1 and R_2 are 11.254kΩ, then

$$b = \frac{1}{R' C \omega_p} = 1.414207$$

3. The expression for $R_6 = \frac{R}{b^2}$; then $R_6 = 5k\Omega$

4. For this case

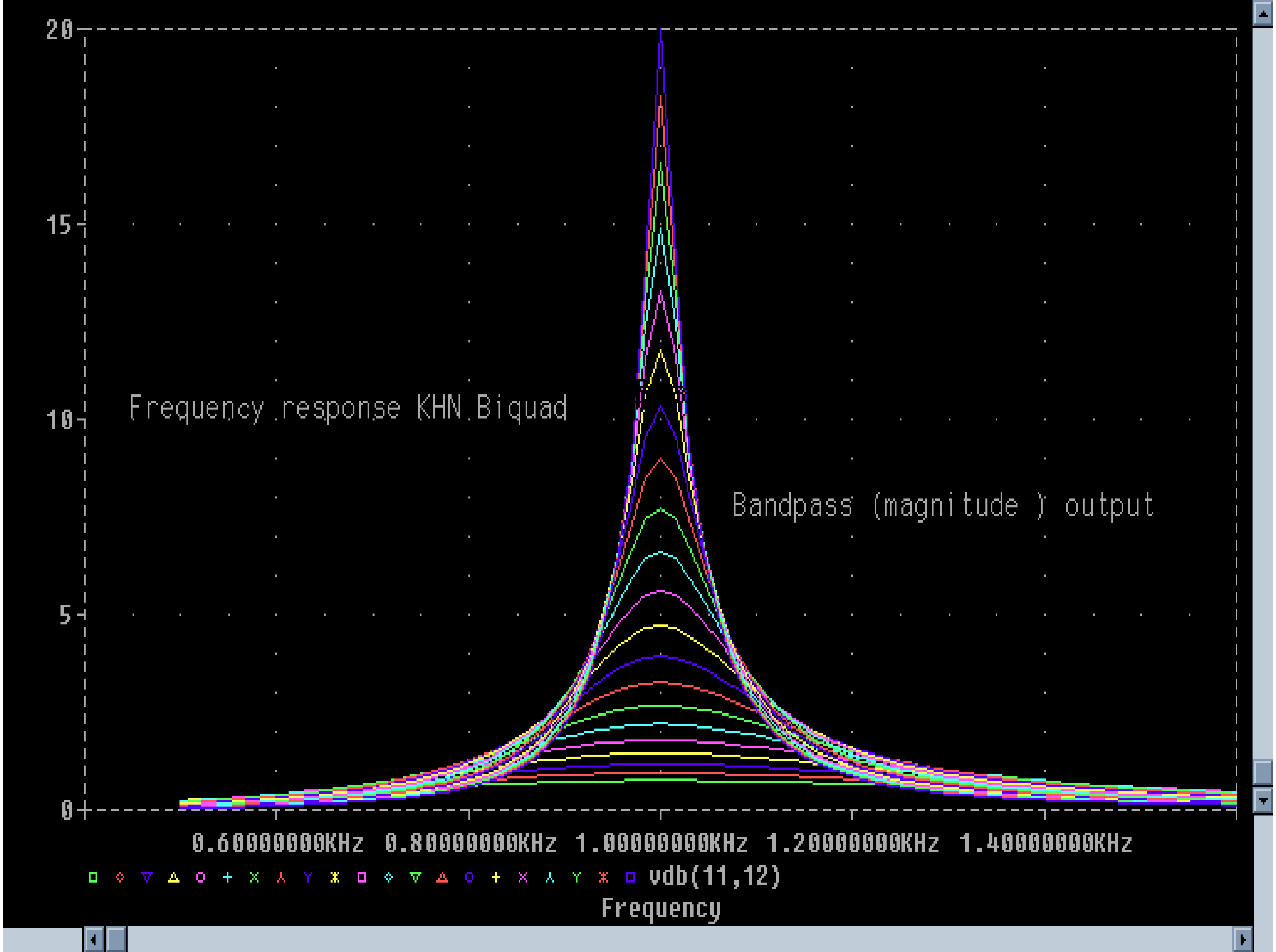
$$R_3 = \frac{R}{H_{BP}} = 3.33\,k\Omega$$

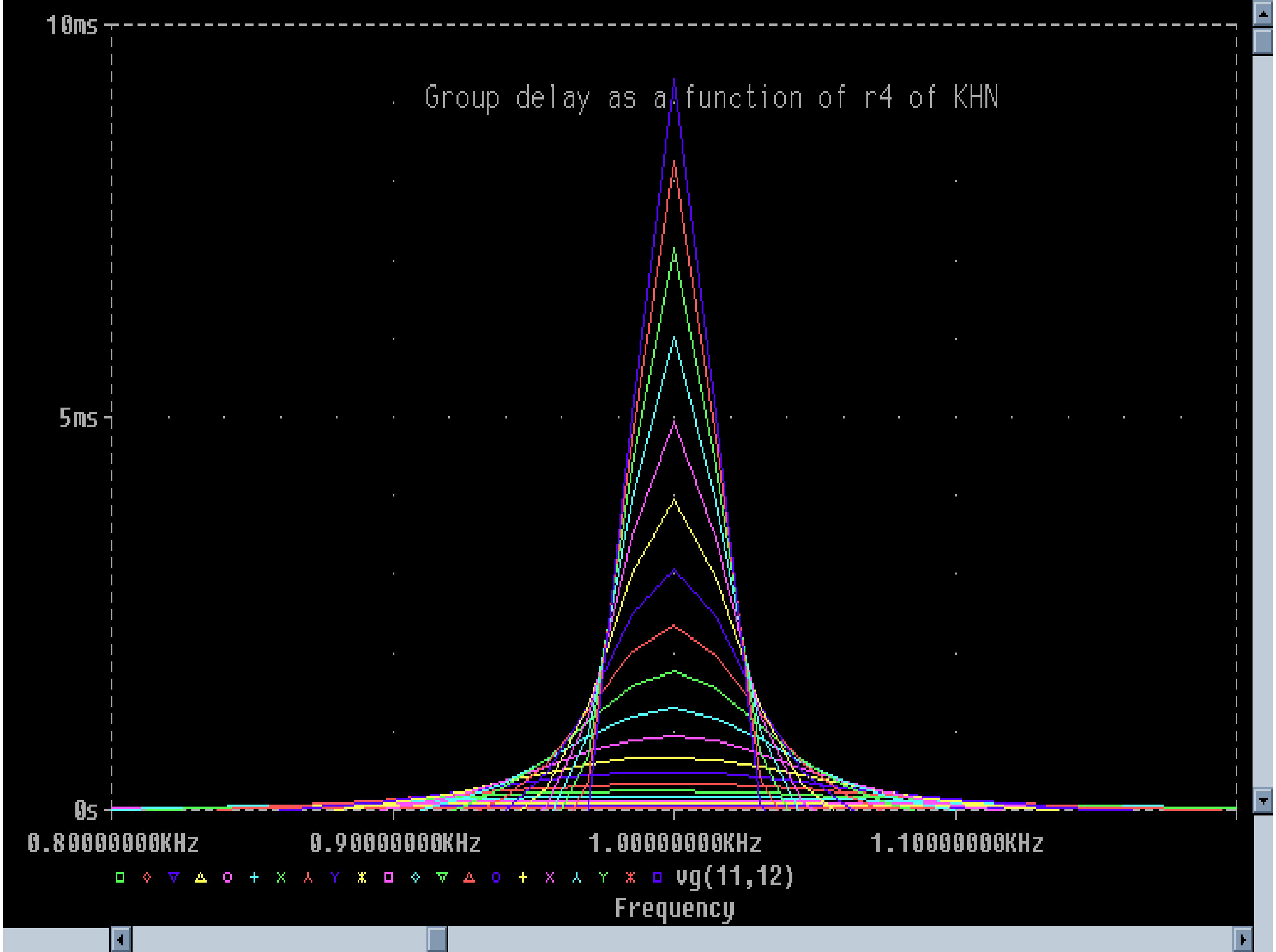
and

$$R_9 = \frac{R}{bQ_p\left[1+\left(1/b^2\right)\right]-1-H_{BP}} = 260\,\Omega$$

Exercise.- Propose component value condition to make this structure Fully Balance Fully Symmetric Structure

```
KERWIN-HUELSMAN-NEWCOMB Biquad Circuit with f0=1KHz, Q=20, Peak
value gain =3
** Description of the passive components
r1      4       3        11.254K
r2      11      5        11.254K
r3      1       12       3.3K
* varying r4 values and parameters with the .step statement
.param R=1
r4      1       11       {R}
r5      2      6  10K
r6      2      3   5K
r9      1       0         260
c7      4       11        0.01U
c8      5       6         0.01U
* Description of Op Amps
E1      3       0       1       2       2D5
E2      11      0       0       4       2D5
E3      6       0       0       5       2D5
*
VIN     12      0       AC      1
*
.AC LIN 100 500        2K
.step DEC param R  300 30K 10
.PLOT AC VDB(11) VP(11)
.PROBE
.END
```


20
15
10
5
0
Frequency response KHN Biquad
Bandpass (magnitude) output
0.60000000KHz 0.80000000KHz 1.00000000KHz 1.20000000KHz 1.40000000KHz
vdb(11,12)
Frequency



Group delay as a function of r4 of KHN
10ms
5ms
0s
0.80000000KHz
0.90000000KHz
1.00000000KHz
1.10000000KHz
vg(11,12)
Frequency

Second-Order Filter Characteristics and Mason Rule to obtain Transfer Functions

- Second-Order Transfer function characteristics.
  - How can you use this information for better design and tuning
  - Time domain measurements and characterization
- Mason Rule is a good tool, borrow from control theory, to easily obtain transfer functions from a signal flow graph

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PROPERTIES of SECOND-ORDER SYSTEMS AND MASON'S RULE to determine transfer functions by inspection

- Mathematical definitions and properties of Second-Order Systems.
- Building block second-order system architectures and properties.
- Mason's Rule to obtain easily transfer functions and to facilitate the generation of new architectures.

Second-Order Filter Types

Second-order blocks are important building blocks since with a combination of them allows the implementation of higher-order filters. The general order transfer function in the s-plane has the form:

$$H(s) = \frac{K_1 s^2 + K_2 s + K_3}{s^2 + \frac{\omega_o s}{Q} + \omega_p^2}$$

Particular conventional cases are:

Lowpass	i.e.,	$K_1 = K_2 = 0$
Bandpass	i.e.,	$K_1 = K_3 = 0$
Highpass	i.e.,	$K_2 = K_3 = 0$
(Notch) Band-Elimination	i.e.,	$K_2 = 0$
Allpass	i.e.,	$K_1 = 1, \quad K_2 = -\frac{\omega_o}{Q} \quad \text{and} \quad K_3 = \omega_o^2$

One interesting case used for amplitude equalization is the “equalizer” sometimes referred to as Bump (DIP) Equalizer. In this case, $K_1 = 1 \quad K_3 = \omega_o^2$ and $K_2 = \pm k \frac{\omega_o}{Q}$.

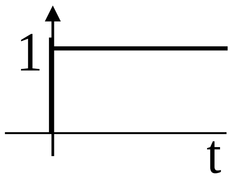
Specific structures have different properties. Some structures have enough degrees of freedom to allow them to change independently ω_o , Q (or BW) and a particular gain $|H(\omega_p)|$ where ω_p is a particular frequency, i.e., $\omega_p = 0, \quad \omega_o, \quad \infty$ for the LP, BP and HP cases. Furthermore, some structures have the property to have constant Q or BW while varying f_o. We will illustrate later, by examples, some of the structures with such properties.

Properties of Second-Order Systems in the time domain

$$s^2 + \frac{\omega_o}{Q}s + \omega_o^2 = (s+\alpha)^2 + \beta^2 = s^2 + 2\alpha s + \omega_o^2$$

where $\quad \alpha = \dfrac{\omega_o}{2Q}, \quad \beta = \omega_o\sqrt{1 - \dfrac{1}{4Q^2}}$

v_{in}

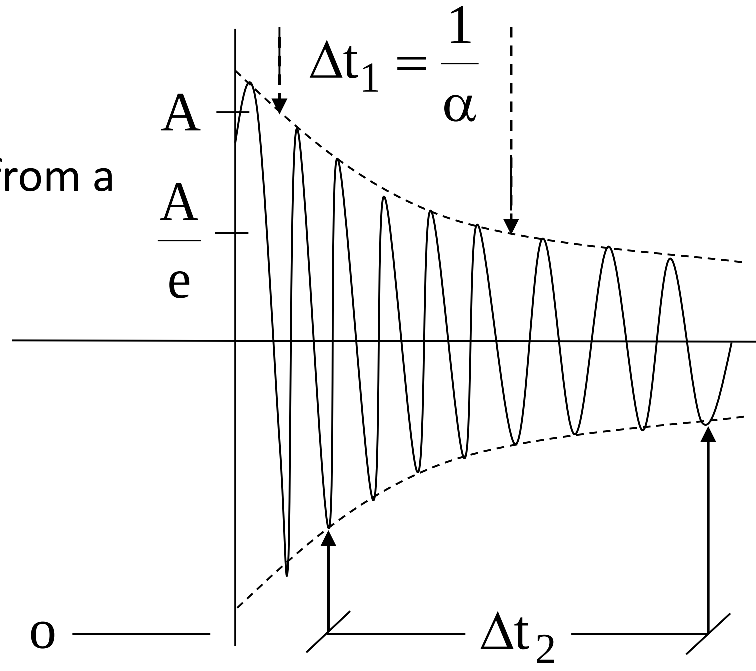


$$\frac{N(s)}{(s+\alpha)+\beta^2}$$

$$v_{o_s}(t) = K_1 + K_2 e^{-\alpha t}\sin(\beta t + \theta)$$

How to determine the pole location from a step response?

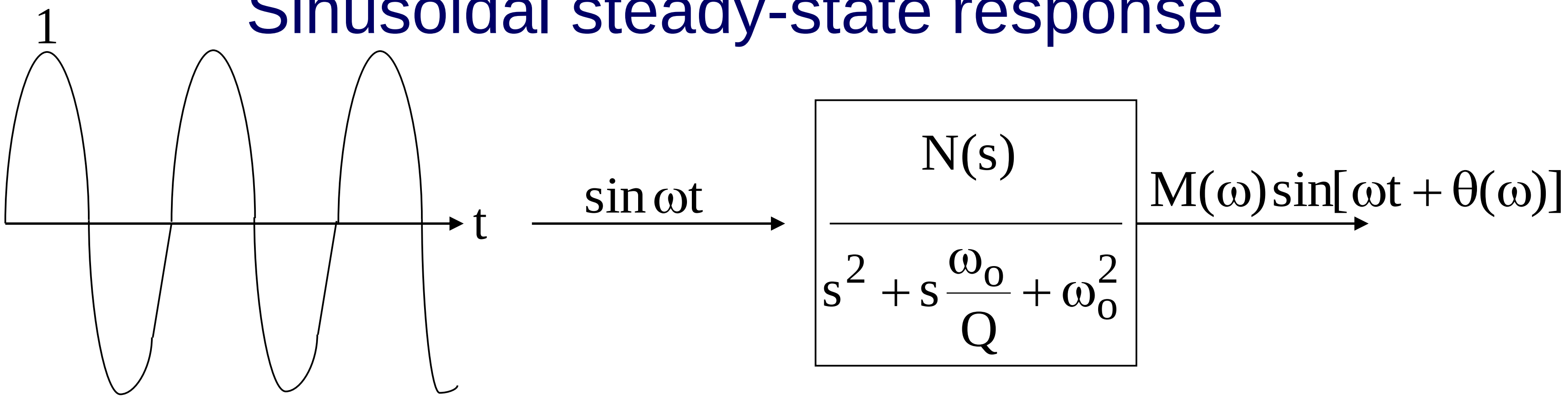
$K_1,\ K_2 = f(N(s))$

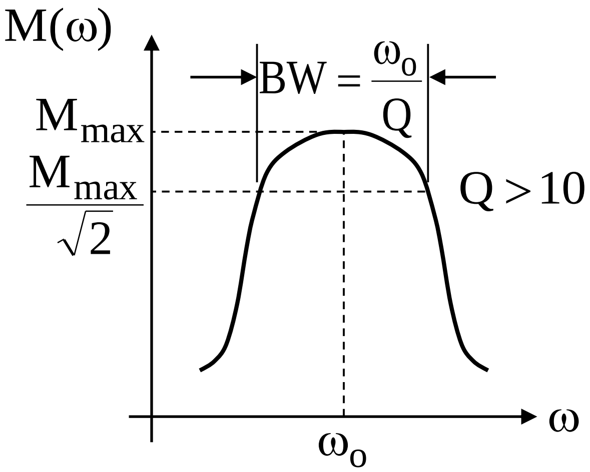


$$\Delta t_2 = \frac{2\pi N}{\beta}$$

N CYCLES

Sinusoidal steady-state response





$$M(\omega) = \left| \left[\frac{N(s)}{s^2 + s\dfrac{\omega_o}{Q} + \omega_o^2} \right]_{s=j\omega} \right| = \frac{|N(j\omega)|}{\sqrt{(\omega_o^2 - \omega^2)^2 + \left(\dfrac{\omega_o}{Q}\omega\right)^2}}$$

Only two measurements are necessary to fix the position of the complex poles. The measurement of the frequency of peaking determines the magnitude of the poles, ω_o, and the measurement of the 3-dB bandwidth determines ω_o / Q.

Second-Order Low-Pass Networks

$$T(s) = \frac{H}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2}$$

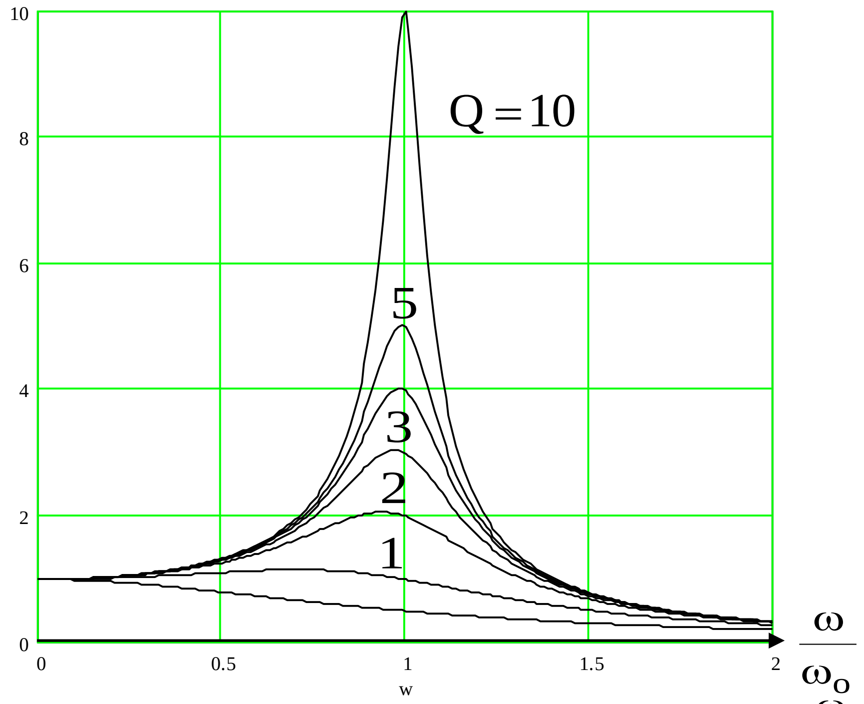
Since H is merely a magnitude scale factor, let $H = \omega_o^2$.

$$T(s) = \frac{\omega_o^2}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2} = \frac{1}{\left(\frac{s}{\omega_o}\right)^2 + \left(\frac{s}{\omega_o}\right)\frac{1}{Q} + 1}$$

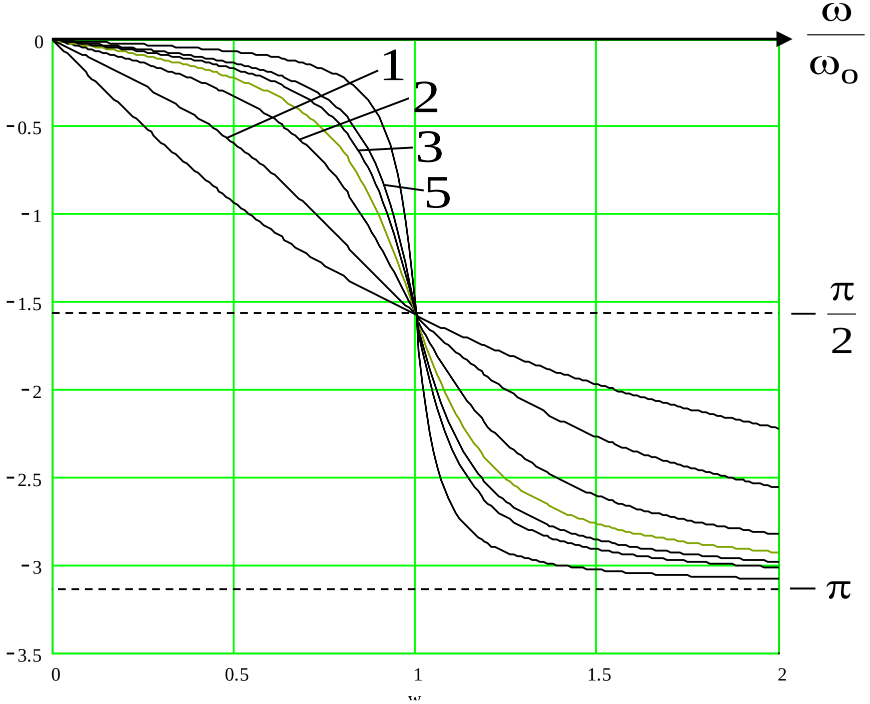
1. For $\omega/\omega_o \ll 1$, $|T(j\omega)| \cong 1$. Therefore, low frequencies are passed.
2. For $\omega/\omega_o \gg 1$, $|T(j\omega)| \cong (\omega_o/\omega)^2$. Therefore, high frequencies are attenuated.
3. For $Q > 1/\sqrt{2}$, the magnitude peaks at $\frac{\omega}{\omega_o} = \sqrt{1 - \frac{1}{2Q^2}}$

   The peak occurs at a frequency lower than ω_o. For Q>5, the frequency of peaking practically equals ω_o(within 1%).
4. For Q>5, the 3-dB bandwidth is practically equal to ω_o/Q rad/s.
5. At $\omega/\omega_o = 1$, $|T(j\omega_o)| = Q$ and the phase is $-\pi/2$.
6. At $\omega/\omega_o \gg 1$, the phase is $-\pi$.
7. For Q>5, the phase undergoes a rapid shift of π radians about ω_o.

Magnitude



Phase



Magnitude and phase of

$$\frac{\omega_o^2}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2}$$

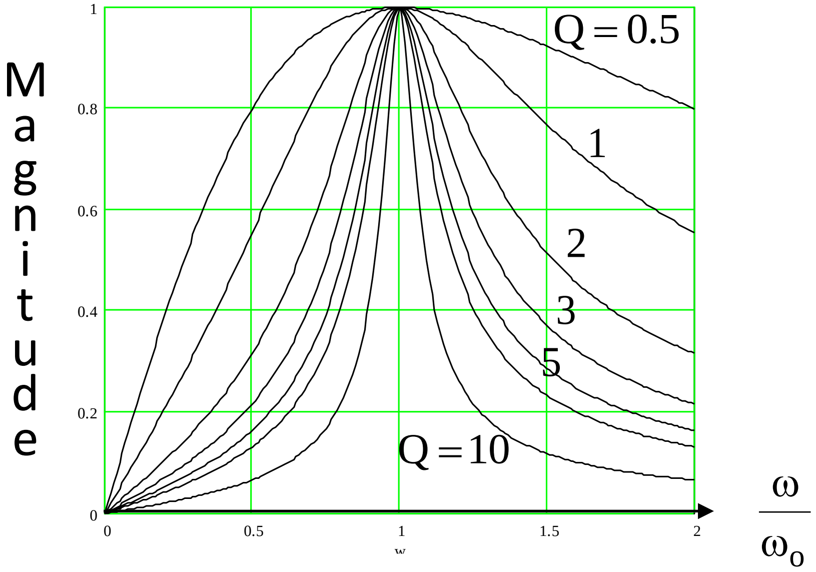
The Second-Order Band-Pass Function

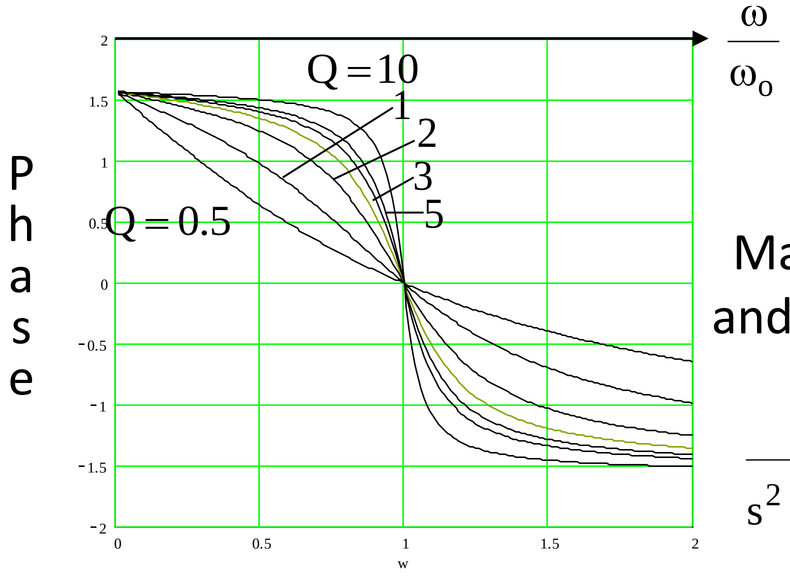
The second-order band-pass function has one zero at the origin and another at infinity:

$$T(s) = \frac{Hs}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2}$$

To normalize the peak value of the magnitude function to unity, let $H = (\omega_o / Q)$:

$$T(s) = \frac{\frac{\omega_o}{Q}s}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2} = \frac{\frac{1}{Q}\frac{s}{\omega_o}}{\left(\frac{s}{\omega_o}\right)^2 + \left(\frac{s}{\omega_o}\right)\frac{1}{Q} + 1}$$



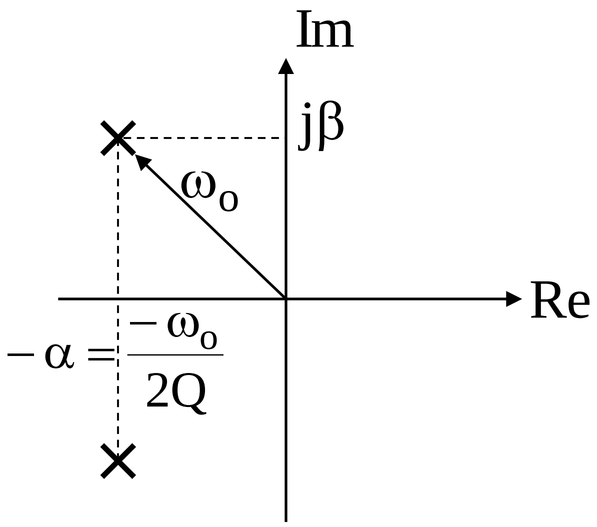


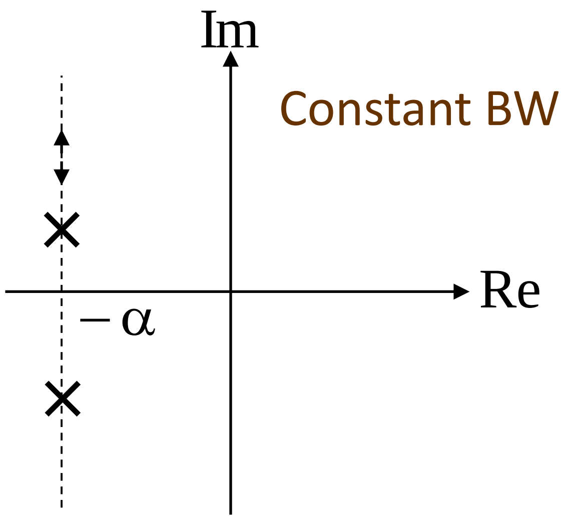
Magnitude and phase of

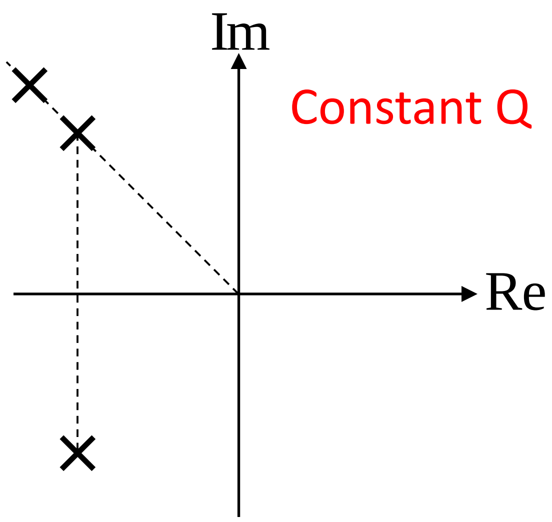
$$\frac{\frac{\omega_o}{Q}s}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2}$$

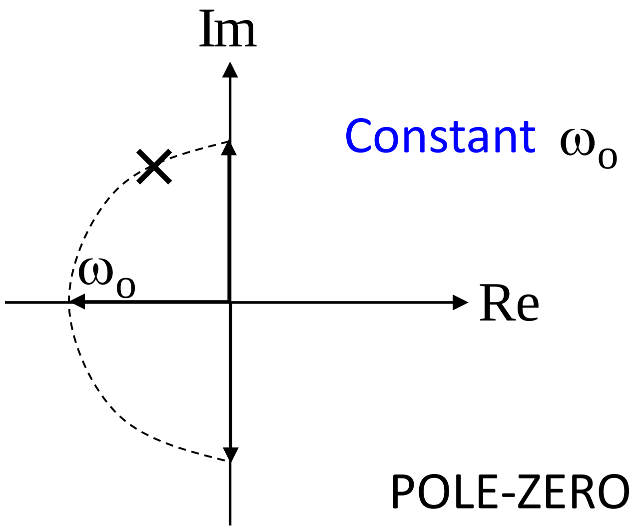
Pole locations and properties

$$\alpha = \frac{1}{\Delta t_1} \quad \text{and} \quad \beta = N\frac{2\pi}{\Delta t_2}$$









POLE-ZERO LOCI

In practical implementations besides the general specifications $(\omega_o, Q, |H(\omega_p)|)$ other particular specifications are imposed which are application dependent. Among them are silicon area, dynamic range, power supply rejection ratio, power consumption, tolerance, accuracy, and sensitivity. This last parameter is often used as a figure of merit. i.e.,

$$S_x^p = \frac{\partial p}{\partial x} \cdot \frac{x}{p} \qquad (1)$$

The above definition is usually referred as normalized sensitivity due to the (*x/p*) factor. *x* and *p* are the variable of the network (i.e. R, C, g_m) and the variable under consideration (i.e., ω_o, Q, $|H(\omega_p)|$)

The General Input-Output Gain Mason Formula

We can reduce complicated block diagrams to canonical form, from which the control ratio is easily written:

$$\frac{V_{out}}{V_{in}} = \frac{G}{1 \pm GH}$$

It is possible to simplify signal flow graphs in a manner similar to that of block diagram reduction. But it is also possible, and much less time-consuming, to write down the input-output relationship by *inspection* from the original signal flow graph. This can be accomplished using the formula presented below. This formula can also be applied directly to block diagrams, but the signal flow graph representation is easier to read - especially when the block diagram is very complicated.

Signal Flow Graphs

Let us denote the ratio of the input variable to the output variable by *T*. For linear feedback control systems, $T=V_{out}/V_{in}$. For the general signal flow graph presented in preceding paragraphs V_{out} is the output and V_{in} is the input.

The general formula for and signal flow graph is

$$T = \frac{\sum_i P_i \Delta_i}{\Delta}$$

where P_i = the *i*th forward path gain

P_{jk} = *j*th possible product of *k* non-touching loop gains

$$\Delta = 1 - (-1)^{k+1} \sum_k \sum_j P_{jk}$$

$$= 1 - \sum_j P_{j1} + \sum_j P_{j2} - \sum_j P_{j3} + \cdots$$

= 1 - (sum of all loop gains) + (sum of all gain-products of 2 non-touching loops) - (sum of all gain-products of 3 non-touching loops + …

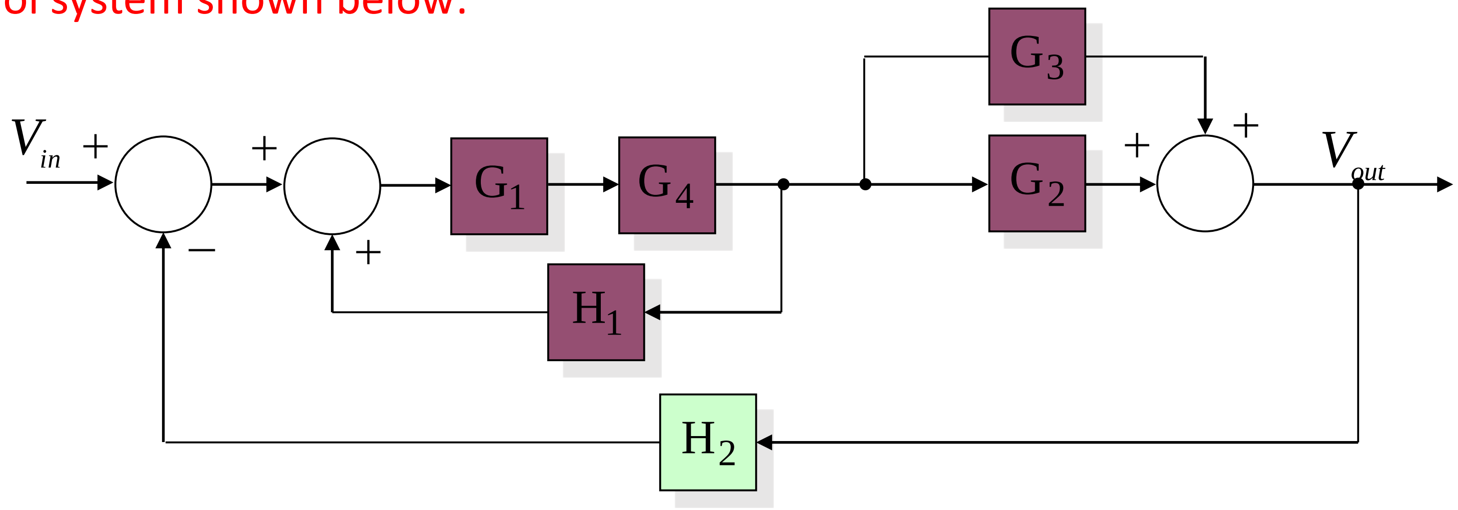
Δ_i = Δ evaluated with all loops touching P_i eliminated.

Two loops, paths, or a loop and a path are said to be **non-touching** if they have no nodes in common.

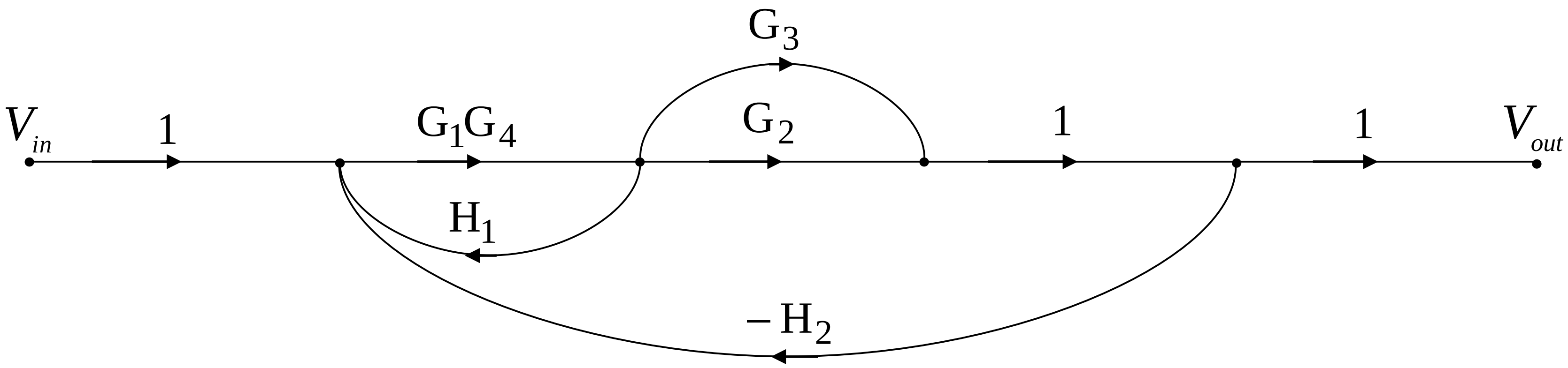
Δ is called the **signal flow graph determinant** or **characteristic function**, since $\Delta = 0$ is the system characteristic equation.

Examples

Let us determine the control ratio Vout/Vin and the canonical block diagram of the feedback control system shown below:



The signal flow graph is



There are two forward paths: $P_1 = G_1G_2G_4, \quad P_2 = G_1G_3G_4$

There are three feedback loops:

$$P_{11} = G_1G_4H_1, \qquad P_{21} = -G_1G_2G_4H_2, \qquad P_{31} = -G_1G_3G_4H_2$$

There are no non-touching loops, and all loops touch both forward paths; then

$$\Delta_1 = 1, \qquad \Delta_2 = 1$$

Therefore the control ratio is

$$T = \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

$$= \frac{G_1G_4(G_2 + G_3)}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

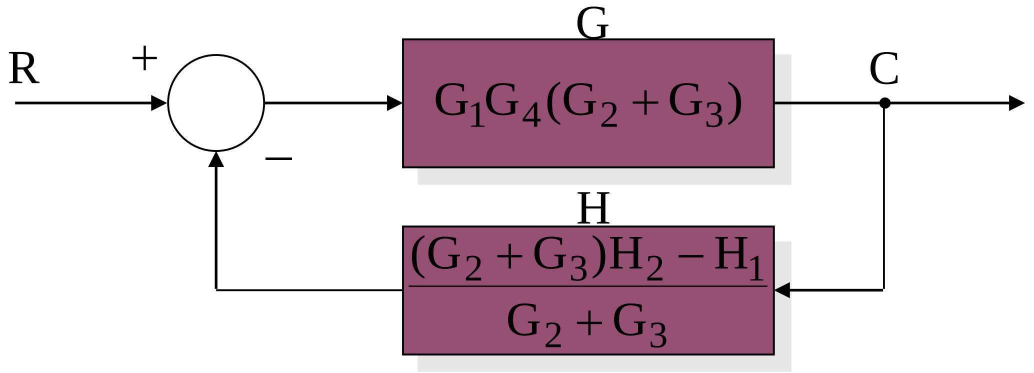
From Equations (8.3) and (8.4), we have

$$G = G_1G_4(G_2 + G_3) \qquad \text{and} \qquad GH = G_1G_4(G_3H_2 + G_2H_2 - H_1)$$

Therefore

$$H = \frac{GH}{G} = \frac{(G_2 + G_3)H_2 - H_1}{G_2 + G_3}$$

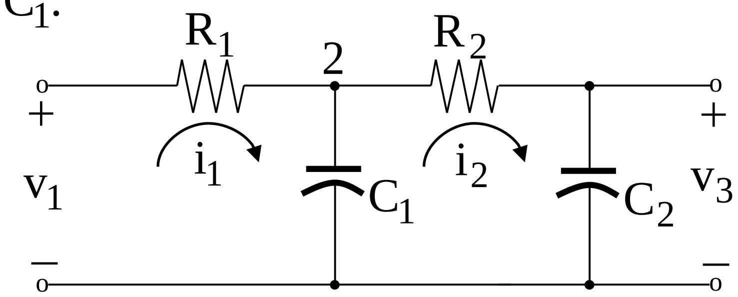
The canonical block diagram is there fore given by



The negative summing point sign for the feedback loop is a result of using a positive sign in the GH formula above.

Example

Draw a signal flow graph for the following resistance network in which $v_2(0) = v_3(0) = 0.$ v_2 is the voltage across C_1.

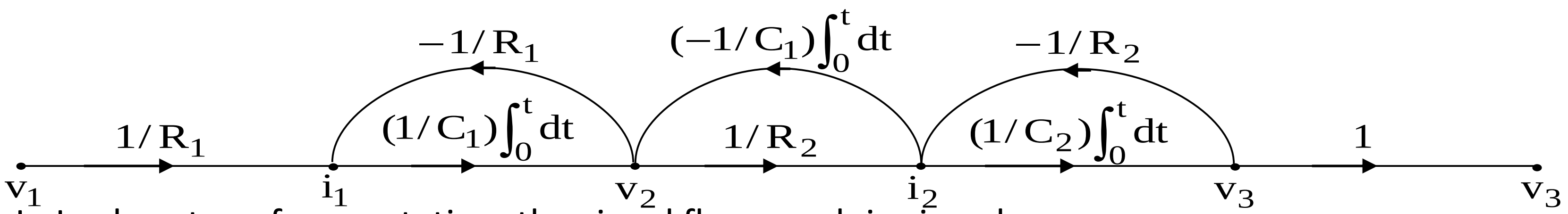


The five variables are $v_1, v_2, v_3, i_1,$ and i_2; and v_1 is the input. The four independent equations derived from Kirchoff's voltage and current laws are

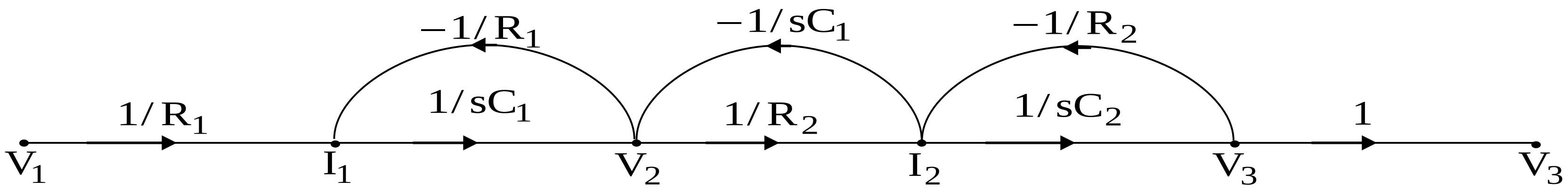
$$i_1 = \left(\frac{1}{R_1}\right)v_1 - \left(\frac{1}{R_1}\right)v_2, \qquad v_2 = \frac{1}{C_1}\int_0^t i_1 dt - \frac{1}{C_1}\int_0^t i_2 dt,$$

$$i_2 = \left(\frac{1}{R_2}\right)v_2 - \left(\frac{1}{R_2}\right)v_3, \qquad v_3 = \frac{1}{C_2}\int_0^t i_2 dt$$

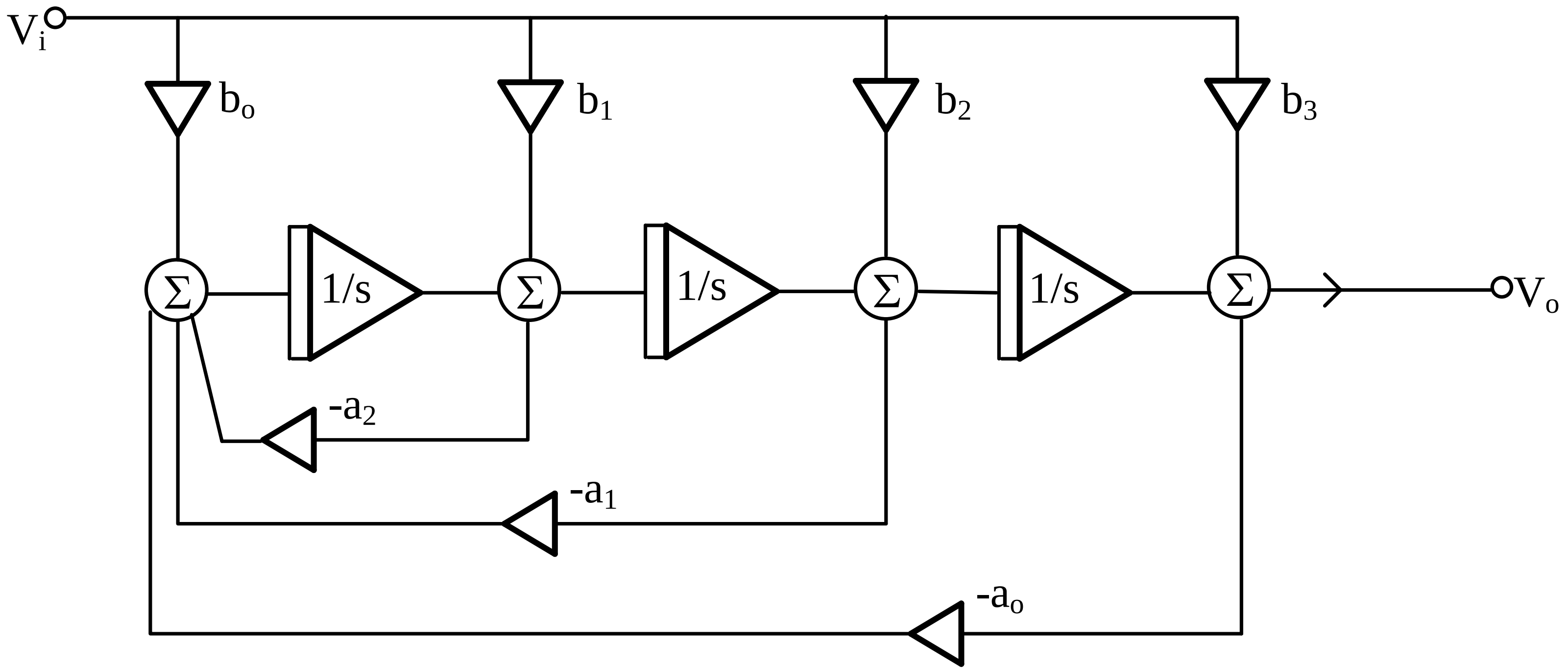
The signal flow graph can be drawn directly from these equations:



In Laplace transform notation, the signal flow graph is given by

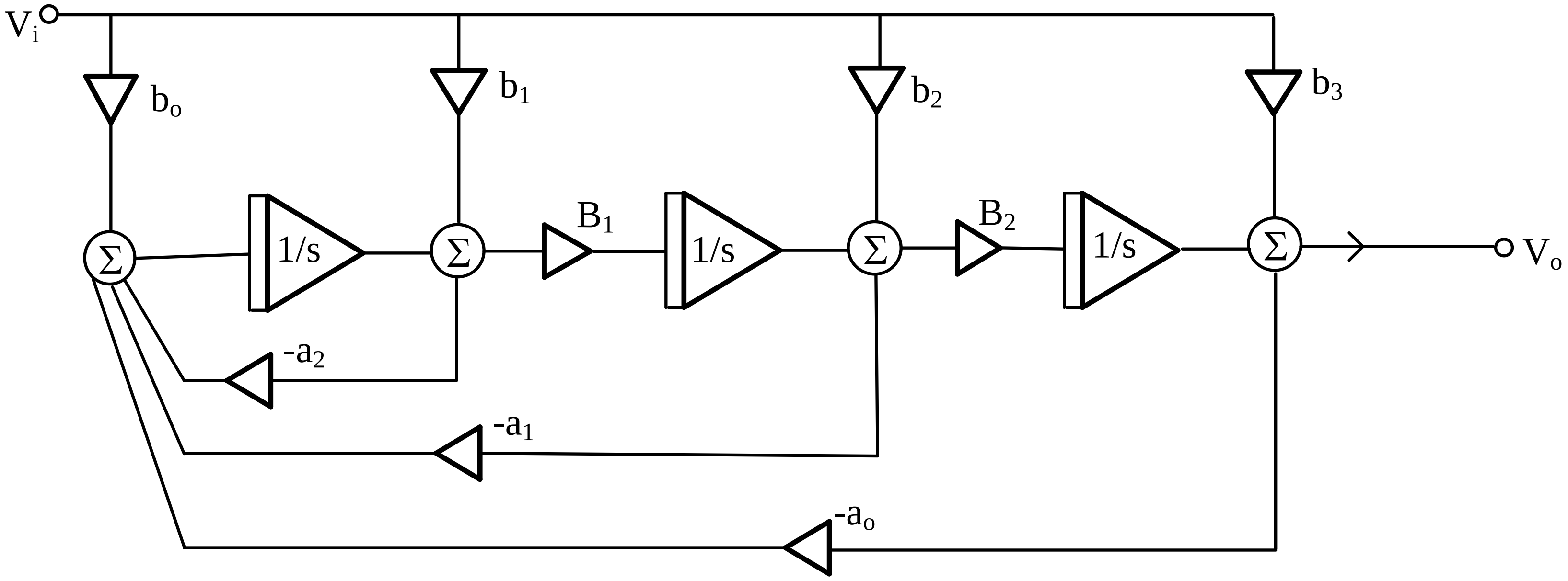


Example of use of Mason Rule in a 3rd Orders State-Variable (observable) Filter



$$H_1(s) = \frac{\frac{b_o}{s^3} + \frac{b_1}{s^2} + \frac{b_2}{s} + b_3}{1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_o}{s^3}} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_o}{s^3 + a_2 s^2 + a_1 s + a_o}$$

What is wrong with this application of Mason's Rule?



$$H_2(s) = \frac{\dfrac{b_o B_1 B_2}{s^3} + \dfrac{b_1 B_1 B_2}{s^2} + \dfrac{b_2 B_2}{s}\left(1 + \dfrac{a_2}{s}\right) + b_3\left(1 + \dfrac{a_2}{s} + \dfrac{B_1 a_1}{s^2}\right)}{1 + \dfrac{a_2}{s} + \dfrac{a_1 B_1}{s^2} + \dfrac{B_1 B_2 a_o}{s^3}}$$

$$H_2 = \frac{b_3 s^3 + \left(b_{32} B_2 + b_3 a_2 + b_1 B_1 B_2\right)s^2 + \left(b_2 B_2 a_2 + b_3 B_1 a_1\right)s + B_1 B_2 b_o}{s^3 + a_2 s^2 + a_1 B_1 s + B_1 B_2 a_o}$$

A possible implementation of H_2 (s) using Active-RC follows