SWITCHED-CAPACITOR FUNDAMENTALS AND CIRCUITS

- Mathematical Background
- Basic Building Blocks
- SC Filters
- SC Oscillators
FUNDAMENTALS ON SC CIRCUITS

\[ \tau = R \cdot C = R_{eq} \cdot C = \frac{T}{C_{eq}} \cdot C = \frac{1}{f_c} \left( \frac{C}{C_{eq}} \right) \]

very accurate!

• Bottlenecks: Switches and High-Speed Op Amps. Non-idealities \( \rightarrow \) enemies!

• The best approach in audio applications.

• Design is carried-out in the Z domain, same domain as for digital circuits.

• Conventional Programming is done via capacitor banks. Spread?

• New non-uniform sampling techniques opens practical application possibilities.

ECEN 622 (ESS)
Examples of Analog and Digital Signals

Analog-to-Digital Converter

\[ x_{in} \xrightarrow{\text{Low-Pass}} x_1 \xrightarrow{\text{Sampling Circuit}} x_2 \xrightarrow{\text{Quantizer}} y_1 \xrightarrow{\text{Decoder}} y_o \]

CT

\[ f_c \]

CT

SD

yo 1’s and 0’s
## Analog and Digital Signals and Systems Concepts

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<th>Types of signals</th>
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<td>-- Discrete values in time</td>
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</table>
OPERATOR $Z^{-1} \iff$ UNIT DELAY

$$Z^{-1}Y(Z) = Z^{-1} \sum_{n=0}^{\infty} y(n)Z^{-n}$$

$$= \sum_{n=0}^{\infty} y(n)Z^{-(n+1)} = \sum_{n=1}^{\infty} y(n-1)Z^{-n}$$

$$Z^{-1}Y(Z) = \sum_{n=0}^{\infty} y(n-1)Z^{-n} - y(-1) = Y[y(n-1)] - y(-1)$$

If $y(nT) = 0$ for $t < 0$, $y(-1) = 0$

$$Z^{-1}Y(Z) = Z^{-1}[y(nT)] = Y[y(n-1)T]$$

In general

$$Z^{-m}Y(Z) = Y[y(n-m)T]$$
STABILITY OF A SECOND-ORDER HOMOGENEOUS DIFFERENCE EQUATION

The C.E.

\[ y(n) + a_1 y(n-1) + a_2 y(n-2) = 0 \]

\[ Y(Z)\left(1 + a_1 Z^{-1} + a_2 Z^{-2}\right) = 0 \]

\[ Z^2 + a_1 Z + a_2 = \left(Z - Z_1\right)\left(Z - Z_2\right) = 0 \]

COMPLEX ROOTS

\[ Z_1, Z_2 = \frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}, \text{ if } \frac{a_1^2}{4} - a_2 < 0, Z_1, Z_2 = re^{\pm j\theta} = x \pm jy \]

\[ r = \sqrt{x^2 + y^2} = \sqrt{a_2}, \cos \theta = \frac{x}{r} = -\frac{a_1}{2\sqrt{a_2}} \]

The C. E. yields

\[ Z^2 - 2r \cos \theta Z + r^2 = 0 \]

Natural Response

\[ y(n) = K r^n \cos\left(2\pi \frac{nf}{fs} + \phi\right) \]

\[ a_1, a_2 \leftrightarrow r, \theta \]

\[ k, \phi \leftarrow \text{Initial Conditions} \]
Relationship between the poles in the Z-Plane and coefficients $a_1$ and $a_2$.

- $a_1^2 > 4a_2$ for Real Roots

\[
\begin{align*}
-a_1 &= 2r \cos \theta \\
a_2 &= r^2
\end{align*}
\]

For Complex Stable Roots.
MAPPING BETWEEN THE S- AND THE Z-PLANES FOR HIGH SAMPLING

Backward

\[ Z_{p1} = \frac{1}{1 - s_p T} \]

Forward

\[ Z_{p2} = 1 + s_p T \]

Bilinear

\[ Z_{p3} = \frac{1 + s_p T / 2}{1 - s_p T / 2} \]

Impulse Invariant

\[ Z_{p4} = e^{s_p T} \]

For small \( s_p T, s_p T << 1 \).

\[ Z_{p1} \approx 1 + s_p T + (s_p T)^2 + (s_p T)^3 + \cdots \]

\[ Z_{p2} = 1 + s_p T \]

\[ Z_{p3} \approx 1 + (s_p T) + \frac{(s_p T)^2}{2} + \frac{(s_p T)^3}{4} + \cdots \]

\[ Z_{p4} \approx 1 + s_p T + \frac{(s_p T)^2}{2} + \frac{(s_p T)^3}{6} + \cdots \]

Ignoring higher order effects

\[ Z_{p1} \approx Z_{p2} \approx Z_{p3} \approx Z_{p4} \approx 1 + s_p T \]

For Very High Sampling Rate

The Approximation For The Mapping is the Same!

\[ Z_p \approx 1 + s_p T \]
RELATIONS BETWEEN POLES
IN THE S- AND Z- DOMAIN

\( s^2 + \frac{\omega_n}{Q} s + \omega_n^2 \)
\( = (s + a)^2 + b^2 = 0 \)

\( a = \frac{\omega_n}{2Q} \)
\( b = \frac{\omega_n}{2Q} \sqrt{4Q^2 - 1} \)

\( Z^2 + b_1Z + b_2 = 0 \)
\( Z^2 - 2r \cos \theta Z + r^2 \)
\( Z_{pi} = 1 + s_{pi} T \)

\( s_{p1,2} = -\frac{\omega_n}{2Q} \pm j\frac{\omega_n}{2Q} \sqrt{4Q^2 - 1} \)
\( s_{p1,2} = -a \pm jb \)

\( Z_{p1,2} = 1 + S_{p1,2} T \)
\( Z_{p1,2} = (1 - aT) \pm jbT = re^{\pm j\theta} \)
\( r = \sqrt{(1 - aT)^2 + (bT)^2} \)
\[ r = \sqrt{\left(1 - \frac{\omega_n T}{2Q}\right)^2 + \left(\omega_n T\right)^2} \approx \sqrt{1 - \frac{\omega_n T}{Q} + \left(\omega_n T\right)^2} \]

OR

\[ Q = \frac{\omega_n T}{1 - r^2 + \left(\omega_n T\right)^2} = \frac{\omega_n T}{(1 + r)(1 - r) + \left(\omega_n T\right)^2} \]

High Q's

\[ Q \bigg|_{r \to 1} \approx \frac{\omega_n T}{2(1 - r) + \left(\omega_n T\right)^2} \bigg|_{\omega_n T << 1} \approx \frac{\omega_n T}{2(1 - r)} \]

then

\[ r = \frac{2Q - \theta}{2Q} = 1 - \frac{\theta}{2Q} \]

Also by a series expansion

\[ \cos \theta \approx 1 - \frac{\theta^2}{2} \]
Thus

\[ b_1 = 2r \cos \theta \approx -2 \left( 1 - \frac{\theta}{2Q} \right) \left( 1 - \frac{\theta^2}{2} \right) \]

\[ b_1 \approx \left( 2 - \theta^2 - \frac{\theta}{Q} \right) \]

\[ b_2 = r^2 \]

\[ b_2 \approx \left( 1 - \frac{\theta}{2Q} \right)^2 = 1 + \frac{\theta^2}{4Q^2} - \frac{\theta}{Q} \]

Also see Table of mappings.
In fact if we let,

\[ b_2 \approx 1 - \frac{\theta}{Q} \]

then

\[ b_1 = (b_2 + 1) + \theta^2 \]

Recall that high Q and high sampling rate conditions have been used. Thus

\[ \theta^2 = 1 + b_2 + b_1 \]

and

\[ Q \approx \frac{\theta}{1 - b_2} \]
MAPPINGS
Forward
Backward
Bilinear
Impulse Invariant
Mixed

\[ H(s) = \frac{N(S)}{s^2 + \frac{\omega_o}{Q} s + \omega_o^2} \quad \rightarrow \quad H(Z) = \frac{N(Z)}{1 + b_1 Z^{-1} + b_2 Z^{-2}} \]

where

\[ b_1 = 2r \cos \theta \]
\[ b_2 = r^2 \]

(See Table of Quadratic Mappings)
Table 2

<table>
<thead>
<tr>
<th>Type of Mapping</th>
<th>( f_0(H_z) )</th>
<th>( Q )</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
</tr>
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<tbody>
<tr>
<td>Forward</td>
<td>( \sqrt{1 + b_1 + b_2} ) ( T )</td>
<td>( \sqrt{1 + b_1 + b_2} ) ( b_1 + 2 )</td>
<td>( -2 + \frac{\omega_0 T}{Q} )</td>
<td>( 1 - \frac{\omega_0 T}{Q} + (\omega_0 T)^2 )</td>
</tr>
<tr>
<td>Backward</td>
<td>( \frac{1 + b_1 + b_2}{b_2} ) ( T )</td>
<td>( -\sqrt{b_2(1 + b_1 + b_2)} ) ( b_1 + 2b_2 )</td>
<td>( \frac{2 + \omega_0 T}{Q} )</td>
<td>( \frac{1}{1 + \frac{\omega_0 T}{Q} + (\omega_0 T)^2} )</td>
</tr>
<tr>
<td>Bilinear, ( a = 2/T ) ( a \equiv \omega_0 \cot (\omega_0 T/2) )</td>
<td>( a \sqrt{1 + b_1 + b_2} ) ( 1 - b_1 + b_2 ) ( T )</td>
<td>( \sqrt{(1 + b_1 + b_2)(1 - b_1 + b_2)} ) ( 2(1 - b_2) )</td>
<td>( \frac{2(\omega_0^2 - a^2)}{a^2 + \omega_0 a + \omega_0^2} )</td>
<td>( \frac{a^2 - \omega_0 a + \omega_0^2}{a^2 + \omega_0 a + \omega_0^2} )</td>
</tr>
<tr>
<td>Impulse invariant</td>
<td>( \sqrt{\left( \frac{\cos^{-1} \left( \frac{1}{2} b_1 \right)}{2b_2} \right)^2 + \frac{1}{4} \left( \ln b_2 \right)^2} ) ( T )</td>
<td>( \sqrt{\left( \frac{\cos^{-1} \left( \frac{1}{2} b_1 \right)}{2b_2} \right)^2 + \frac{1}{4} \left( \ln b_2 \right)^2} )</td>
<td>( \frac{\omega_0 T}{Q} \cos \left( \frac{\omega_0 T}{2Q} \sqrt{4Q^2 - 1} \right) )</td>
<td>( -\frac{\omega_0 T}{Q} )</td>
</tr>
</tbody>
</table>

\( b_1 = -2r \cos \theta \)
\( b_2 = r^2 \)
| Type of Integrator | Magnitude, $|H(e^{j\omega T})|$ | Phase, $\text{Arg } H(e^{j\omega T})$ | Mapping (Equivalent) | Transfer Function |
|-------------------|-----------------|----------------|----------------------|------------------|
| ![Inverting Integrator](image1) | $\frac{\omega_o}{\omega}$ | $\frac{\pi}{2}$ | In the S-Plane | $i.e. H(s) = -\frac{1}{sR_1C_2} = -\frac{\omega_o}{s}$ |
| ![Non-Inverting Integrator](image2) | $\frac{\omega_o}{\omega \sin(\omega T/2)}$ | $\frac{\pi}{2}$ | LDI | $H(z) = -\frac{C_1}{C_2} \frac{1}{1 - z^{-1}}$ |
| ![Inverting Integrator (Backward)](image3) | $\frac{\omega_o}{\omega \sin(\omega T/2)}$ | $\frac{-\pi}{2}$ | LDI | $H(z) = -\frac{C_1}{C_2} \frac{1}{1 - z^{-1}}$ |
| ![Inverting Integrator (Forward)](image4) | $\frac{\omega_o}{\omega \sin(\omega T/2)}$ | $\frac{-\pi - \omega T}{2}$ | Forward | $H(z) = -\frac{C_1}{C_2} \frac{1}{1 - z^{-1}}$ |
Type of Filters in the Z-Domain

Specifications

\[ H(s) = \frac{\pm H_0 \omega_p^2}{s^2 + \frac{\omega_p}{Q} s + \omega_p^2} \]

i.e.,

consider

\[ H(s) = \frac{\pm H_0 \omega_p^2}{s^2 + \frac{\omega_p}{Q} s + \omega_p^2} \]

Use prewarping

Apply: A backward transformation

\[ s \rightarrow \left(1 - Z^{-1}\right)/T \]

then

\[ H(z) = \frac{\pm K_o Z^2}{r^2 - 2r \cos \theta Z + Z^2} \]

: A forward Transformation

\[ H(z) = \frac{\pm K_o}{r^2 - 2r \cos \theta Z + Z^2} \]

: A Bilinear Transformation

\[ H(z) = \frac{K_o (1 + Z)^2}{r^2 - 2r \cos \theta Z + Z^2} \]
b) $N[z] = \pm K_o Z^2$

c) $N(z) = \pm K_o$

d) $N(z) = (1 + Z)^2$
Another Example.

Bandpass

\[ H(s) = \frac{\pm H_0 \omega_p}{Q} \frac{\omega_p}{Q} s + \omega_p^2 \]

\[ s^2 + \frac{\omega_p}{Q} s + \omega_p^2 \]

i) Backward Transformation

\[ s \rightarrow \left(1 - Z^{-1}\right)/T \]

\[ H(z) = \frac{\pm H_0 \frac{\omega_p}{QT} (1 - Z^{-1})}{(1 - Z^{-1})^2} + \frac{\omega_{pi} 1 - Z^{-1}}{Q_i T} + \omega_{pi}^2 \]

\[ H(z) = \frac{\omega_p T Z (Z - 1)}{Q(1 + \omega_p T/Q + (\omega_p T)^2)} \]

\[ \frac{1}{1 + \frac{\omega_p T}{Q}} \left(1 + \frac{\omega_p T}{Q}\right)^2 - Z \frac{Z + \omega_p T}{1 + \frac{\omega_p T}{Q} + (\omega_p T)^2} + Z^2 \]

\[ \frac{1}{1 + \frac{\omega_p T}{Q}} + (\omega_p T)^2 - Z \]

\[ \frac{Z + \omega_p T}{1 + \frac{\omega_p T}{Q} + (\omega_p T)^2} + Z^2 \]

BACKWARD

FORWARD

BILINEAR
<table>
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<th>TYPE OF FILTER</th>
<th>NUMERATOR</th>
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<td>LP 20 (bilinear)</td>
<td>$K_o (1 + z^{-1})^2$</td>
</tr>
<tr>
<td>LP 02 (forward)</td>
<td>$K_o z^{-2}$</td>
</tr>
<tr>
<td>LP11</td>
<td>$K_o z^{-1} (1 + z^{-1})$</td>
</tr>
<tr>
<td>LP 10</td>
<td>$K_o (1 + z^{-1})$</td>
</tr>
<tr>
<td>LP 01</td>
<td>$K_o z^{-1}$</td>
</tr>
<tr>
<td>BP 20 (bilinear)</td>
<td>$K_o (1 - z^{-1})(1 + z^{-1})$</td>
</tr>
<tr>
<td>BP 01 (forward)</td>
<td>$K_o z^{-1}(1 - z^{-1})$</td>
</tr>
<tr>
<td>BP 00 (backward)</td>
<td>$K_o (1 - z^{-1})$</td>
</tr>
<tr>
<td>HP</td>
<td>$K_o (1 - z^{-1})^2$</td>
</tr>
<tr>
<td>Notch (symmetric)</td>
<td>$K_o [1 + 2(1 - \cos \Theta_O)z^{-1} + z^{-2}]$</td>
</tr>
</tbody>
</table>

where $\cos \Theta_O = \frac{2r \cos \Theta}{1 + r^2}$

$\Theta_O$ and $\Theta$ are the angles of the zero and pole locations, respectively.

Same as above with $\cos \Theta_O > \frac{2r \cos \Theta}{1 + r^2}$

Same as above with $\cos \Theta_O < \frac{2r \cos \Theta}{1 + r^2}$

$K_o \left( r^2 - 2r \cos \Theta z^{-1} + z^{-2} \right)$

Table Numerators of second-order transfer functions in the z-plane

$D(s) = 1 - 2r \cos \Theta z^{-1} + r^2 z^{-2}$
Examples of Basic Implementations

- First-order Low Pass (Continuous-Time)

\[
H(s) = \frac{K}{1 + s/\omega_p} \quad ; \quad \frac{V_o(s)}{V_{in}(s)} = \frac{K\omega_p}{s + \omega_p} \quad ; \quad \frac{dv_o(t)}{dt} + \omega_p v_o(t) = K\omega_p v_{in}(t)
\]

Continuous-time

\[
C = \frac{1}{R\omega_p}
\]

Stability implies to have poles in the left-half plane (LHP)
\[ H(s) = H(j\omega) = \frac{-K}{1 + j\omega/\omega_p} = \frac{K}{\left[1 + \left(\frac{\omega}{\omega_p}\right)^2\right]^{1/2}} \angle -\tan \frac{\omega}{\omega_p} \]

\[ s = j\omega \]

Frequency Response

\[ h(t) = K\omega_p e^{-\omega_p t} \]

Impulse Response
First-Order Low-Pass (Discrete-Time) Sampled-Data (i.e., Switched-Capacitor)

\[ H(z) = \frac{V_o(z)}{V_{in}(z)} = \frac{K_1}{1 - z^{-1}/\omega_{pl}} = \frac{K_1 \omega_{pl}}{\omega_{pl} - z^{-1}}; \]

\[ [\omega_{pl} - z^{-1}]V_o(z) = K_1 \omega_{pl} V_{in}(z) \]

\[ \omega_{pl} V_o(z) - z^{-1}V_o(z) = K_1 \omega_{pl} V_{in}(z) \]

Taking the inverse z-transform

\[ \omega_{pl} v_o(nT) - v_o(n-1)T = K_1 \omega_{pl} v_{in}(nT) \]

\[ v_o(n-1)T - \omega_p v_o(nT) = -K_1 \omega_{pl} v_{in}(nT) \]

This difference equation represents the first-order low pass in the Z-domain.

Note that

\[ Z^{-1}X_o(z) \Rightarrow x_o(n-1)T \]

\[ Z^{-b}X_o(z) \Rightarrow x_o(n-b)T \]
Frequency Response (A periodic transfer function!)

\[
H(z) = \frac{K_1 \omega_p Z}{\omega_p Z - 1} = \frac{K_1 \omega_p \left(\cos \omega T + j\sin \omega T\right)}{\left(\omega_p \cos \omega T - 1\right) + j\sin \omega T}
\]

\[
z = e^{j\omega}
\]

\[
H(e^{j\omega}) = \frac{K_1 \omega_p}{\left(\omega_p \cos \omega T - 1\right)^2 + \sin^2 \omega T} \left[\omega T - \tan^{-1} \left(\frac{\sin \omega T}{\omega_p \cos \omega T - 1}\right)\right]
\]

where

\[
K_1 \omega_p = \frac{C_1}{C}
\]

\[
\omega_p = 1 + \frac{C_2}{C}
\]
What are the relationships between the s-plane and z-plane?

• There are a number of mappings between the two planes

• The most popular and exact is the bilinear mapping.

\[
s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{2}{T} \frac{z - 1}{z + 1} \quad \text{or} \quad z = \frac{1 + \left(\frac{T}{2}\right)s}{1 - \left(\frac{T}{2}\right)s}
\]

• The commonly used for high-sampling rate is:

\[
s = \frac{1}{T} \frac{1 - z^{-1}}{z^{-1}} = \frac{1}{T} \frac{z - 1}{1}
\]

or

\[
z = sT + 1
\]
Example. Approximate a first-order low-pass continuous-time to a discrete-time low-pass under high-sampling conditions. \((f_s/f > > 1)\)

\[
H(s) = \frac{K}{1 + \frac{s}{\omega_p}} = \frac{K\omega_p}{s + \omega_p} \quad \left| s = \frac{1}{T}(z - 1) \right.
\]

\[
H(z) = \frac{K\omega_p T}{z - 1 + \omega_p T} = \frac{K\omega_p T}{z - (1 - \omega_p T)}
\]

What is the 3dB cut-off frequency in both domains?

\[
f_{3dB} = \omega_p \quad \text{CT}
\]

For \(H(z)\) is more complex than the computation of \(f_{3dB}\).

\[
H(e^{j\omega T}) = \frac{K\omega_p T}{\cos \omega T + j\sin \omega T - (1 - \omega_p T)}
\]

\[
|H(e^{j\omega T})| = \left|H(e^{j\omega_{3dB} T})\right|/\sqrt{2}
\]

\[
\omega = \omega_{3dB}
\]

\[
\omega_{3dB} = \frac{1}{T} \cos^{-1} \left(\frac{2 - 2\omega_p T - (\omega_p T)^2}{2(1 - \omega_p T)}\right)
\]
Numerical example. Low-Pass (First Order)

\( \omega_p = 2\pi \times 10^3 \text{ r/s} \)

\( f_c = f_s = 20\text{KHz} \quad ; \quad T = 1/f_s \quad ; \quad \omega_p T = \frac{2\pi \times 10^3}{20 \times 10^3} = 0.1\pi \)

K = 2

For the continuous-time

\[ f_{3\text{dB}} = 1\text{KHz} \]

\[ H(j\omega) \bigg|_{\omega = \omega_{3\text{dB}}} = \frac{2}{\sqrt{2}} = 1.4142 \]

For the sample-data

\[ f_{3\text{dB}} = \frac{f_s}{2\pi} \cos^{-1} \left( \frac{2 - 2\omega_p T - (\omega_p T)^2}{2(1 - \omega_p T)} \right) \]

\[ f_{3\text{dB}} \approx 1.16\text{KHz} \]
Switched - Capacitor Filters

• Use Z-transform mathematics

• Are described by difference equations

• Time constants are proportional to capacitor ratios

• Best implementation for audio applications

• Originally the basic goal was to replace resistors by switches and capacitors

• This design approach is one of the most popular in the industry
SC Advantages

- Reduced silicon area
- Good accuracy. Time constants are implemented with capacitor ratios (~0.1%)
- Don’t require a low-impedance output stage (OTA’s could be used)
- Could be implemented using digital circuit process technology
- Very useful in the audio range
**NOTATION**

Switches

Representation

Network

With No Switches

\( \phi_{1,3} \)

\( \phi_{1,4,n} \)

EXAMPLE \( n=2 \)

**Phase Period Representation**

\[
\begin{align*}
0 & \quad \frac{T}{n} & \quad 2\frac{T}{n} & \quad 3\frac{T}{n} & \quad \cdots & \quad \frac{(n-2)T}{2} & \quad \frac{(n-1)T}{2} & \quad T \\
\hline
1 & \quad 2 & \quad 3 & \quad \cdots & \quad \frac{n-1}{n} & \quad n
\end{align*}
\]

Clock Period

**NON-OVERLAPPING**

**PHASE PERIODS OF A CONVENTIONAL CLOCK SEQUENCE**
S/H and its respective odd and even components

\[ y_H(t) = y_{H_1}^o(t) + y_{H_1}^e(t) \]

or

\[ Y_H(Z) = Y_{H_1}^o(Z) + Y_{H_1}^e(Z) \]
CONVENTIONAL NOTATION FOR TRANSFER FUNCTION IS:

\[ H(Z) = \frac{V_o(Z)}{V_{in}(Z)} \]

where \( i \) and \( j \) can be either “e” or “o”. i.e.,

\[ H^{ij}(Z) = \frac{V_{o}^{j}(Z)}{V_{in}^{i}(Z)} \]

\[ H^{eo}(Z) = \frac{V_{o}^{o}(Z)}{V_{in}^{e}(Z)} \]
TWO-PHASE CLOCK GENERATOR

SINGLE PHASE

CLOCK SIGNALS

NON-OVERLAPPING
SWITCHED-CAPACITOR EQUIVALENT RESISTOR

Continuous (Conventional) Resistor

\[ R = \frac{V_1 - V_2}{I_1} = \frac{V_2 - V_1}{I_2} \Rightarrow I_2 = \frac{V_2 - V_1}{R} \]

\( V_1 \) and \( V_2 \) are constant voltage sources

\[ \begin{align*}
  i_1 &\quad \phi_1 \\
  i_2 &\quad \phi_2 \\
\end{align*} \]

\[ \begin{align*}
  \phi_1 &\quad T/2, T, 3T/2, 2T \\
  \phi_2 &\quad T/2, T, 3T/2, 2T \\
\end{align*} \]
At time $t = t_0$ we apply the clocks.

At $\phi_1$

\[
Q(t_o + T/2) = CV_1
\]

At $\phi_2$

\[
Q(t_o + T) = CV_2 - CV_1
\]

For the next period, at $\phi_1$

\[
Q(t_o + \frac{3T}{2}) = C(V_1 - V_2)
\]

\[
i = \frac{dQ}{dt}
\]

\[
Q_1 = \int_{t_o + T}^{t_o + 3T/2} i_1(t)dt = \int_{t_o + T/2}^{t_o + 3T/2} i_1(t)dt
\]
The average current $I_{1\,(\text{aver})}$ becomes

$$I_{1\,(\text{aver})} = \frac{Q(t_o + \frac{3T}{2})}{T}$$

or

$$I_{1\,(\text{aver})} = \frac{1}{T} \int_{t_o + \frac{3T}{2}}^{t_o + \frac{3T}{2} + \Delta t} i_1(t) \, dt$$

$$I_{1\,(\text{aver})} = \frac{C(V_1 - V_2)}{T} = \frac{\Delta Q}{\Delta t}$$

Comparing with the continuous time resistor

$$R_{eq} = \frac{T}{C} = \frac{1}{f_C C}$$

EXAMPLE. $R = 250\, \text{K}\Omega$, $f_C = 128\, \text{KHz}$

$$\frac{A_R}{A_C} \approx 32$$

$$C = \frac{1}{R_{eq} f_C} = \frac{1}{250 \times 128 \times 10^6} = 31.25\, \text{pF}$$

$\text{AREA}$ $\quad 5,776 \quad 178.57 \quad \text{mils}^2$
ACCURACY OF TIME CONSTANTS

**CONTINUOUS TIME:**

\[ \tau = R_1 C_2 \]

\[ \frac{d\tau}{\tau} = \frac{dR_1}{R_1} + \frac{dC_2}{C_2} \rightarrow \pm 40\% \rightarrow \pm 65\% \]

where \( \frac{d\tau}{\tau} \) is interpreted as the accuracy of \( \tau \).

**TEMPERATURE DEPENDANT!**

**DISCRETE TIME:**

\[ \tau = \frac{1}{f_{CC_1}} \cdot C_2 = T\left(\frac{C_2}{C_1}\right) \]

\[ \frac{d\tau}{\tau} = \frac{dT}{T} + \frac{dC_2}{C_2} - \frac{dC_1}{C_1} \]

\[ \frac{d\tau}{\tau} \approx \frac{dC_2}{C_2} - \frac{dC_1}{C_1} \rightarrow 0.1\% \]
**KCL KIRCHHOFF “CHARGE” LAW**

\[ \sum_{i=1}^{n} Q_i = 0 \]

\[ Q_j = C_j V_{C_j} \]

\[ V_{x} \quad \frac{a}{+} \quad V_{C_j} \quad \frac{-}{b} \quad V_{y} \]

at \( t = t_2 \)

\[ V_{C_j} = V_x(t_2) - V_y(t_2) - (V_x(t_1) - V_y(t_1)) \]

Voltage across the capacitor \( V_{C_j} \) = voltage difference (at present time) across the capacitor minus initial condition (at past time).
CHARGE CONSERVATION
ANALYSIS METHOD

\[ q_L(n) = q_M(n - 1) + \dot{q}_C(n) \]

\[
\begin{align*}
\phi_1 & \quad \phi_2 \\
n - 1 & \quad n - \frac{1}{2} & n & \quad n + \frac{1}{2}
\end{align*}
\]

EXAMPLE

for \( \phi_2 \)

\[ C_2[V_2^e(n) - V_2^o(n - 1/2)] + C_1[V_2^e(n) - V_1^o(n - 1/2)] = 0 \]

for \( \phi_1 \)

\[ C_2[V_2^o(n - 1/2) - V_2^e(n - 1)] = 0 \Rightarrow V_2^o(n - 1/2) = V_2^e(n - 1) \]
THEN

\[ C_1 v_2^0(n) + C_2 v_2^0(n) = C_2 v_2^0(n - 1) + C_1 v_1^0(n - 1) \]

\[ (C_1 + C_2)V_2^0(Z) - C_2 Z^{-1}V_2^0(Z) = C_1 V_1^0(Z)Z^{-1} \]

\[ H^{oo}(Z) = \frac{C_1 Z^{-1}}{C_1 + C_2 - \frac{C_2}{C_1} Z^{-1}} = \frac{Z^{-1}}{1 + \alpha - \alpha Z^{-1}} \]

\[ H^{oo}(Z) = \frac{C_1 Z^{-1}}{C_1 + C_2 - \frac{C_2}{C_1} Z^{-1}} = \frac{Z^{-1}}{1 + \alpha - \alpha Z^{-1}} \]

\[ H^{oo}(Z) = \frac{V_0^0(Z)}{V_{in}^0(Z)} = \frac{V_0^e(Z)Z^{-1/2}}{V_{in}^0(Z)} \]
\[ V_2^c(n) = V_2^c(n + \frac{1}{2}) \]

\[ C_2[V_2^0(n + \frac{1}{2}) - V_2^0(n - \frac{1}{2})] + C_1[V_2^0(n + \frac{1}{2}) - V_1^0(n - \frac{1}{2})] \]

\[ C_2[V_2^0(Z)]Z^{\frac{1}{2}} - Z^{-\frac{1}{2}} + C_1V_2^0(Z)Z^{\frac{1}{2}} = C_1V_1^0(Z)Z^{-\frac{1}{2}} \]

\[ C_2(1 - Z^{-1})V_2^0(Z) + C_1V_2^0(Z) = C_1V_1^0(Z)Z^{-\frac{1}{2}} \]

\[ \frac{V_2^0(Z)}{V_1^0(Z)} = \frac{C_1Z^{-1}}{C_2(1 - Z^{-1}) + C_1} = \frac{C_1Z^{-1}}{C_2 + C_1 - C_2Z^{-1}} \]

\[ \frac{V_2^0(Z)}{V_1^0(Z)} = \frac{C_1}{(C_2 + C_1)Z - C_2} = \frac{C_1}{(C_2 + C_1) + (C_2 + C_1)ST - C_2} \]

\[ Z \approx 1 + ST \]

For high-sampling rate \( \omega T \ll 1 \)

\[ \frac{V_2^0(Z)}{V_1^0(Z)} = \frac{C_1}{C_1 + (C_2 + C_1)5T} = \frac{1}{1 + \frac{(C_2 + C_1)5T}{C_1}} = \frac{1}{1 + \frac{5}{(C_2 + C_1)T}} \]

\[ Z \approx 1 + ST \]

\[ f^3_{3d} \approx \frac{1}{2\pi} \cdot \frac{C_1}{(C_2 + C_1)T} = \frac{1}{2\pi} \cdot \frac{f_c}{1 + \frac{C_2}{C_1}} \]

Aside:

\[ f^3_{3d} \approx \frac{1}{2\pi} \cdot \frac{1}{R_1C_2} \approx \frac{1}{2\pi} \cdot \frac{1}{TC_2} = \frac{1}{2\pi} \cdot \frac{C_2}{C_1} \]
Switched-Capacitor (SC) Filters

How to design SC Filters?

- Two basic approaches

**Filter Specifications**

**Approximation Methods in the s-plane**

- \( H(s) \rightarrow H(z) \) mapping
- Select a RC-Active Filter Topology

**Pick a SC Filter topology, and use coefficient matching to determine capacitor values**

**Use a high-sampling rate and substitute R’s by SC resistors**

**SC Filter Implementation**
Systematic SC Filter Design

1. Design specifications
2. Block, circuit diagrams
   - Transfer function
   - Design equations
3. SC Filter Design
4. Simulate Filter
   - (SWITCAP, ASIZ, etc.)
5. Additional specifications
6. Design amplifiers, switches, etc.
7. Verify design

- doesn’t meet specs
- does meet specs

- Simulate SC at transistor level (if possible)
- Layout circuit
- Extract layout
- Fabricate
- Test
What are the advantages and disadvantages of the two filter design procedures?

• Mapping Techniques
  + Systematic and well documented (see Matlab)
  + It can use any sampling rate, including the (minimum) Nyquist rate.
  - Difficult to implement by hand calculation

• Transforming R to SC resistors
  + It is, conceptually, easier to follow for analog designer
  + Its design is straightforward
  - Yields not an optimal design for area, imposed high sampling rate involves larger capacitor ratios.
Switched-Capacitor Filters Components

**Basic Elements**

**Capacitors**
- polysilicon
- metal1-metal2
- parasitics, clock-feedthrough

**Switches**
- N-MOS
- Transmission gates
- Noise and on-resistance

**OTAS**
- DC-gain
- settling time (GBW, phase margin)
- noise

Non-overlapping clock phases

Typical switched-capacitor integrator
Switched-Capacitor Filters: CAPACITORS

- CP1, CP2” are very small (1-5 % of C1)
- CP2’ is around 10-30 % of C1
SWITCHES

MOS transistors biased in triode region, ~100-100 kΩ

Off resistances ~ G Ω

N-MOS

- Transmission gates

\[ \text{Resistance } R_s \approx \frac{L}{\mu_n C_{ox} W (V_{GS} - V_T)} \]

- Small resistance for low voltages but high resistance for large voltages

- Rail to rail operation, provided that VDD and IVSSI > VT

- Smaller resistance (fast response)

- 2 clock phases

- More parasitics
Switched-Capacitor Filters: SWITCHES

Transmission gates

\[ R_n \approx \frac{L}{\mu_n C_{ox} W(VDD - V1 - V_T)} \]

For \( V1, V2=0, \ W/L=1, \ \mu_n C_{OX}=10^{-4} \)

\[ R_n = 10k/(VDD-1) \]

If \( VDD, |VSS|<V_T \), the resistance is extremely high!!!
Switches (continues)

- CGS and CGD are Linear polysilicon capacitors, introduce offset voltage.

- CSB, DB are non-linear capacitors, introduce harmonic distortion components.

- Mobile charge introduces gain errors and harmonic distortion components.

\[
C_{GS, GD} = C_{OX}WL_D + \frac{1}{2}C_{OX}WL
\]

\[
C_{BS, BD} = \frac{1}{2} \frac{C_{j0}WL}{\sqrt{1 + \frac{2\phi_F}{VSB}}} + C_{jBS}
\]

\[
\text{mobile charge} = C_{OX}WL(V_G - V_T)
\]
Operational Transconductance Amplifier

**PRACTICAL CONSIDERATIONS:**

- **DC-gain.** The inverting input is not a real virtual ground. \( v_{\text{invert}} = \frac{v_o}{A_{dc}} \)

- **Settling time.** \( C_1 \) must be discharged during phase 1. The main limitation is due to limited output current and phase margin.

- **Clock feedthrough.** Can be alleviated by using especial clocking schemes.

- **Noise.** In most of the practical cases the dominant noise components are due to the Switches!!!
Switched-Capacitor Integrator Analysis

**Charge conservation Principle:**
Charge injected by C1 is equal to the charge absorbed by CF ($\Delta Q_{C1} = \Delta Q_{CF}$)

$$\Delta V_0 CF = -\Delta Q_{C1}$$
Switched-Capacitor Stray-Sensitive Integrator Analysis

Charge conservation Principle:
Charge injected by C1 is equal to the charge absorbed by CF ($\Delta Q_{C1}=\Delta Q_{CF}$)

$$[(0-0)C_1-(v_i(t_0+T/2)-0)C_1]+[(0-v_0(t))C_f-(0-v_0(t_0+T/2))C_f]=0$$

Solving for phase 2 ($t=t_0+T$)

$$H(z) = -\frac{C_1}{C_f} \frac{z^{-1/2}}{1-z^{-1}}$$
Switched-Capacitor Integrators: Stray-Insensitive Integrators

Charge conservation ==> Phase 1
\[(v_i(nT)-0)C_{\text{in}} + [(v_0(nT)-0)C_f - (v_0(nT-T/2)-0)C_f] = 0\]

Phase 2
\[0 = (v_0(nT-T/2)-0)C_f - (v_0(nT-T)-0)C_f\]

Solving for phase 1
\[v_i(nT)C_{\text{in}} - [v_0(nT) - v_0(nT-T)]C_f = 0\]

\[H(z) = \frac{C_{\text{in}}}{C_f} \frac{1}{1 - z^{-1}}\]

Charge conservation ==> Phase 1
\[(0 - v_i(NT-T/2))C_{\text{in}} + [(v_0(nT))C_f - (v_0(nT-T/2))C_f] = 0\]

Phase 2
\[0 = C_f v_0(nT-T/2) - C_f v_0(nT-T)\]
\[v_{\text{cin}}(NT-T/2) = v_i(NT-T/2)\]
\[v_i(nT-T/2)C_{\text{in}} - [v_0(nT) - v_0(nT-T)]C_f = 0\]

\[H(z) = \frac{C_{\text{in}}}{C_f} \frac{z^{-1/2}}{1 - z^{-1}}\]