ECEN 605

LINEAR SYSTEMS

Lecture 3

Laplace Transform II

Solution of LTI Systems by Laplace Transforms

In this section we describe how the Laplace transform can be used to solve linear differential equations with constant coefficients. Equations of this type arise in linear time invariant system (LTI systems) relating the output $y(t)$ to the input $u(t)$, $t \geq 0$.
Consider an LTI system

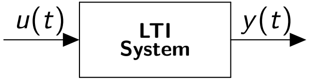

Figure: LTI system

Solution of LTI Systems by Laplace Transforms (cont.)

where $y(t)$ and $u(t)$ are related by

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y$$
$$= b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u. \qquad (1)$$

Solution of LTI Systems by Laplace Transforms (cont.)

Given the known input

$$u(t), \quad t \geq 0 \tag{2}$$

with zero initial conditions at $t = 0^-$, the output

$$y(t), \quad t \geq 0 \tag{3}$$

can be determined if the initial conditions, denoted by $\mathbf{y}(0^-)$

$$\mathbf{y}(0^-) := \left[y(0^-), \dot{y}(0^-), \ddot{y}(0^-), \ldots, y^{n-1}(0^-)\right] \tag{4}$$

are known.

Solution of LTI Systems by Laplace Transforms (cont.)

Taking the Laplace transform of (1) and using the notation $y(t) \leftrightarrow Y(s)$, $u(t) \leftrightarrow U(s)$, we have

$$\begin{aligned} a_n & \left[s^n Y(s) - s^{n-1} y(0^-) - s^{n-2} \dot{y}(0^-) - \cdots - y^{n-1}(0^-)\right] \\ & + a_{n-1}\left[s^{n-1} Y(s) - s^{n-2} y(0^-) - \cdots - y^{n-2}(0^-)\right] \\ & + \cdots + a_1 \left[sY(s) - y(0^-)\right] + a_0 Y(s) \\ = {} & b_m s^m U(s) + b_{m-1} s^{m-1} U(s) + \cdots + b_1 sU(s). \end{aligned} \tag{5}$$

Introduce the polynomials

$$A(s) := a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{6}$$

$$B(s) := b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \tag{7}$$

$$\begin{aligned} P(s, \mathbf{y}(0^-)) := {} & a_n \left[s^{n-1} y(0^-) + s^{n-2} \dot{y}(0^-) + \cdots + y^{n-1}(0^-)\right] \\ & + a_{n-1}\left[s^{n-2} y(0^-) + s^{n-3} \dot{y}(0^-) + \cdots + y^{n-2}(0^-)\right] \\ & + \cdots + a_1 y(0^-). \end{aligned} \tag{8}$$

Solution of LTI Systems by Laplace Transforms (cont.)

Now solving for $Y(s)$ from (5) we obtain

$$Y(s) = \underbrace{\frac{P\left(s, \mathbf{y}(0^-)\right)}{A(s)}}_{Y_0(s)} + \underbrace{\frac{B(s)}{A(s)}U(s)}_{Y_u(s)} \tag{9}$$

or

$$Y(s) = Y_0(s) + Y_u(s). \tag{10}$$

Using the notation $Y_0(s) \leftrightarrow y_0(t)$, $Y_u(s) \leftrightarrow y_u(t)$ we have, from (10), taking inverse Laplace transforms and using linearity of the inverse transform,

$$y(t) = y_0(t) + y_u(t). \tag{11}$$

In (11) we see that the **total** response $y(t)$ is the sum of $y_0(t)$ which depends only on the initial conditions $\mathbf{y}(0^-)$, $\mathbf{u}(0^-)$, (see (9)) and $y_u(t)$ which depends only on the input $u(t)$, $t > 0$.

Solution of LTI Systems by Laplace Transforms (cont.)

Therefore $y_0(t)$ is called the **initial condition response** and $y_u(t)$ is called **forced response**. Alternatively $y_0(t)$ is also called the **zero input response** and $y_u(t)$ is the **zero state response**, that is, the response to $u(t)$ under zero initial conditions. Finally, $\frac{B(s)}{A(s)} =: G(s)$ is called the **system transfer function**, and the roots of $A(s)$ and $B(s)$ are called the **poles** and **zeros** of the system.

Solution of LTI Systems by Laplace Transforms (cont.)

Example

Consider the $R\,L\,C$ circuit.

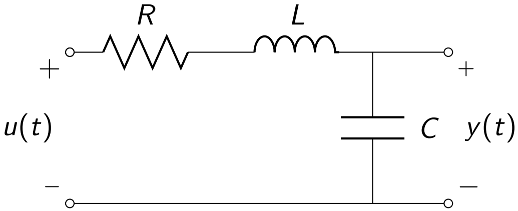

Figure: An RLC circuit

Solution of LTI Systems by Laplace Transforms (cont.)

Example (cont.)

Assuming zero output current (also known as the no loading condition)

$$u(t) = R\,C\,\dot{y}(t) + L\,C\,\ddot{y}(t) + y(t). \tag{12}$$

Taking the Laplace transform of (12), we get

$$\begin{aligned} U(s) &= R\,C\left[s\,Y(s) - y(0^-)\right] \\ &\quad + L\,C\left[s^2\,Y(s) - s\,y(0^-) - \dot{y}(0^-)\right] + Y(s) \end{aligned} \tag{13}$$

so that

$$Y(s) = \underbrace{\frac{R\,Cy(0^-) + L\,C\,s\,y(0^-) + L\,C\,\dot{y}(0^-)}{A(s)}}_{Y_0(s)} + \underbrace{\frac{B(s)}{A(s)}U(s)}_{Y_u(s)} \tag{14}$$

Solution of LTI Systems by Laplace Transforms (cont.)

Example (cont.)

where

$$A(s) = s^2\, L\, C + s\, R\, C + 1$$
$$B(s) = 1 \tag{15}$$

and the system transfer function is:

$$G(s) = \frac{1}{s^2\, L\, C + s\, R\, C + 1}. \tag{16}$$

For given initial conditions $\mathbf{y}(0^-)$ and a specific input $u(t)$, $t \geq 0$ one must take the inverse Laplace transform of (14) to obtain the response $y(t)$. This is discussed next.

Solution for System Response

In this section we describe the procedure to calculate the system response $y(t)$ of an LTI system using Laplace transforms.

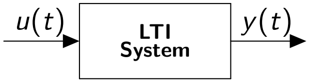

Figure: Input-Output Block Diagram

From the previous section we know that

$$Y(s) = Y_0(s) + G(s)U(s) \tag{17}$$

where $Y_0(s)$ depends only on the initial conditions $\mathbf{y}(0^-)$ and $\mathbf{u}(0^-)$. We proceed through examples.

Solution for System Response (cont.)

Example

Consider the $R\,L\,C$ circuit

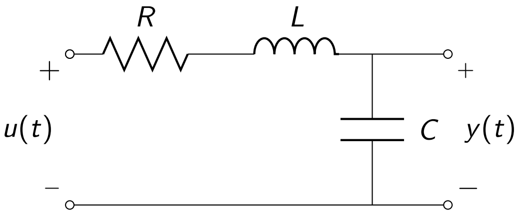

Figure: An RLC circuit

$$u(t) = RC\dot{y}(t) + LC\ddot{y}(t) + y(t). \tag{18}$$

Solution for System Response (cont.)

Example (cont.)

Taking the Laplace transforms

$$\begin{aligned} U(s) = R\,C\left(s\,Y(s) - y(0^-)\right) \\ + L\,C\left(s^2\,Y(s) - s\,y(0^-) - \dot{y}(0^-)\right) \\ + Y(s) \end{aligned} \tag{19}$$

so

$$Y(s) = \frac{R\,C\,y(0^-) + L\,C\,s\,y(0^-) + L\,C\,\dot{y}(0^-)}{A(s)} + G(s)U(s) \tag{20}$$

Suppose that $y(0^-) = 1$, $\dot{y}(0^-) = 1$, $L\,C = 1$ and $R\,C = 1$, then (14) becomes

$$Y(s) = \frac{s+2}{s^2+s+1} + \frac{1}{s^2+s+1}\,U(s). \tag{21}$$

Solution for System Response (cont.)

Example (cont.)

The zero input response is

$$\begin{aligned}
y_0(t) &= \mathcal{L}_-^{-1}\left\{\frac{s+2}{s^2+s+1}\right\} \\
&= \mathcal{L}_-^{-1}\left\{\frac{s+2}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\} \\
&= \mathcal{L}_-^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\} + \mathcal{L}_-^{-1}\left\{\frac{\frac{3}{2}\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\} \\
&= e^{-\frac{1}{2}t}\, cos\frac{\sqrt{3}}{2}t + \sqrt{3}\, e^{-\frac{1}{2}t}\, sin\frac{\sqrt{3}}{2}t. \qquad (22)
\end{aligned}$$

Solution for System Response (cont.)

Example (cont.)

The zero state response is

$$y_u(t) = \mathcal{L}_-^{-1}\Big\{ \underbrace{\frac{1}{s^2+s+1}\,U(s)}_{Y_u(s)} \Big\}. \tag{23}$$

For $u(t) = U(t)$ (unit step), $U(s) = \dfrac{1}{s}$ and

$$Y_u(s) = \frac{1}{s(s^2+s+1)}. \tag{24}$$

Solution for System Response (cont.)

Example (cont.)

Expanding by partial fractions

$$\frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{B\,s+C}{s^2+s+1} = \frac{(A+B)s^2+(A+C)s+A}{s(s^2+s+1)} \qquad (25)$$

so that by equating numerator coefficients in (25)

$$\begin{aligned} A &= 1 \\ A+C &= 0 \quad (C=-1) \\ A+B &= 0 \quad (B=-1). \end{aligned} \qquad (26)$$

Solution for System Response (cont.)

Example (cont.)

Now

$$Y_u(s) = \frac{1}{s} + \frac{-s-1}{s^2+s+1}$$
$$= \frac{1}{s} - \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} - \frac{\frac{1}{2}\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \quad (27)$$

and therefore

$$y_u(t) = U(t) - e^{-\frac{1}{2}t} cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} sin\frac{\sqrt{3}}{2}t. \quad (28)$$

Thus the total response $y(t)$ is the sum $y_0(t) + y_u(t)$ and is given by

$$y(t) = U(t) + \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) e^{-\frac{1}{2}t} sin\frac{\sqrt{3}}{2}t. \quad (29)$$

Solution for System Response (cont.)

Example

Consider the $R\,C$ circuit

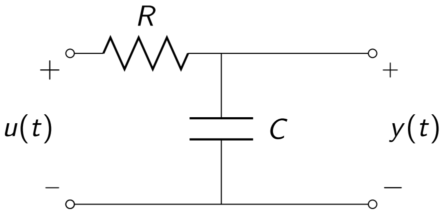

Figure: An RC circuit I

Solution for System Response (cont.)

Example (cont.)

The differential equation of the circuit is (voltage summation around the loop)

$$u(t) = R\,C\,\dot{y}(t) + y(t). \tag{30}$$

By taking Laplace transforms of (30)

$$U(s) = R\,C\,\left[s\,Y(s) - y(0^-)\right] + Y(s) \tag{31}$$

and

$$Y(s) = \underbrace{\frac{R\,C\,y(0^-)}{1 + R\,C\,s}}_{Y_0(s)} + \underbrace{\frac{1}{1 + R\,C\,s}\,U(s)}_{Y_u(s)}. \tag{32}$$

Solution for System Response (cont.)

Example (cont.)

Suppose $R\,C = 1$ and $u(t) = U(t)$, a unit step. Then

$$Y(s) = \frac{y(0^-)}{s+1} + \frac{1}{s(s+1)}. \tag{33}$$

Expanding the second term by partial fractions

$$Y(s) = \frac{y(0^-)}{s+1} + \frac{1}{s} - \frac{1}{s+1} \tag{34}$$

and therefore

$$y(t) = e^{-t}\,y(0^-) + U(t) - e^{-t} \tag{35}$$

is the total response.

Solution for System Response (cont.)

Example (cont.)

If the input is instead an impulse, $u(t) = \delta(t)$ so that $U(s) = 1$, we have

$$Y(s) = \frac{y(0^-)}{s+1} + \frac{1}{s+1} \tag{36}$$

and

$$y(t) = e^{-t}y(0^-) + e^{-t}. \tag{37}$$

Two important differences between the solutions in (35) and (37) are the facts

Solution for System Response (cont.)

Example (cont.)

(A), that in (35)

$$\lim_{t \to \infty} y(t) = 1 \tag{38}$$

whereas as in (37)

$$\lim_{t \to \infty} y(t) = 0 \tag{39}$$

(B), that in (35)

$$y(0^+) = y(0^-) \tag{40}$$

whereas in (37)

$$y(0^+) = y(0^-) + 1. \tag{41}$$

In general the presence of impulses in the input can cause discontinuities in the initial conditions. This is why the $\mathcal{L}_-$ transform is preferred because $y(0^-)$ is known a priori but $y(0^+)$ is not.

Solution for System Response (cont.)

Example

Consider the $R\,C$ circuit again with a different output, namely $y(t)$ is now the voltage across the resistor:

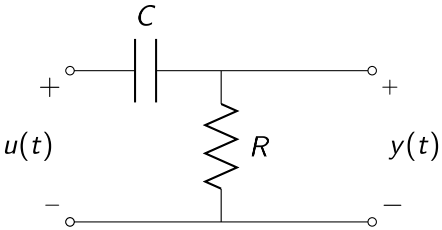

Figure:

Solution for System Response (cont.)

Example (cont.)

The system differential equation is:

$$y(t) = R\,C\,\frac{d}{dt}\,(u(t) - y(t)) = R\,C\,\dot{u}(t) - R\,C\,\dot{y}(t) \tag{42}$$

or

$$R\,C\,\dot{y}(t) + y(t) = R\,C\,\dot{u}(t). \tag{43}$$

Therefore

$$R\,C\,\left[s\,Y(s) - y(0^-)\right] + Y(s) = R\,C\,\left[s\,U(s) - u(0^-)\right] \tag{44}$$

and

$$Y(s) = \underbrace{\frac{R\,C\,[y(0^-) - u(0^-)]}{R\,C\,s + 1}}_{Y_0(s)} + \underbrace{\frac{R\,C\,s}{R\,C\,s + 1}\,U(s)}_{Y_u(s)}. \tag{45}$$

Solution for System Response (cont.)

Example (cont.)

Now suppose $R\,C = 1$, and the input is a unit step. Then $u(0^-) = 0$ and

$$Y(s) = \frac{y(0^-)}{s+1} + \underbrace{\frac{s}{s+1}}_{G(s)}\,\frac{1}{s}. \tag{46}$$

Therefore

$$y(t) = e^{-t}y(0^-) + e^{-t}. \tag{47}$$

Solution for System Response (cont.)

In (47) we note that there is no constant component in the output, even though there is a constant input. Indeed we see that in (46), the numerator of the transfer function cancels the denominator originating from the step. This is a general property of zeros. They **block** signals with poles located at the zeros from passing through to the output.
Additionally we note from (47) that:

$$y(0^+) = y(0^-) + 1 \tag{48}$$

that is, $y(t)$ is discontinuous at $t = 0$. Here, even though $u(t)$ does not contain an impulse, the right hand side of (43) does contain an impulse ($\dot{u}(t)$) when $u(t)$ is a step.

Inverse Laplace transform using partial fractions

In the last section we saw that the response of an LTI system can be calculated by determining the inverse Laplace transform of a rational function. In this section we discuss how this inverse can be found by partial fraction expansion.
Suppose

$$R(s) = \frac{N(s)}{D(s)} \tag{49}$$

denotes a rational function with $N(s)$, $D(s)$ being polynomials with real coefficients and

$$\begin{aligned} \text{degree } N(s) &= m \\ \text{degree } D(s) &= n. \end{aligned} \tag{50}$$

If

$$m < n \tag{51}$$

Inverse Laplace transform using partial fractions (cont.)

$R(s)$ is said to be **strictly proper**. If

$$m = n \tag{52}$$

$R(s)$ is said to be **proper** and if

$$m > n \tag{53}$$

$R(s)$ is **improper**. In general, $R(s)$ can always be rewritten as

$$R(s) = \overline{R}(s) + P(s) \tag{54}$$

where $\overline{R}(s)$ is strictly proper and $P(s)$ is a polynomial of degree $n - m$ (≥ 0.)

Inverse Laplace transform using partial fractions (cont.)

Example

$$R(s) = \frac{s^3 + 2s + 1}{s+1} \tag{55}$$
$$= \underbrace{\frac{A}{s+1}}_{\bar{R}(s)} + B + C\,s + D\,s^2$$
$$= \frac{D\,s^3 + (C+D)\,s^2 + (B+C)\,s + A + B}{s+1}$$

so that

$$D = 1, \quad C + D = 0, \quad B + C = 2, \quad A + B = 1 \tag{56}$$

Inverse Laplace transform using partial fractions (cont.)

Example (cont.)

and thus

$$R(s) = \frac{-2}{s+1} + 3 - s + s^2. \tag{57}$$

Therefore

$$\begin{aligned} \mathcal{L}_-^{-1}\left\{R(s)\right\} &= \mathcal{L}_-^{-1}\left\{\overline{R}(s)\right\} + \mathcal{L}_-^{-1}\left\{P(s)\right\} \\ &= -2\,e^{-t} + 3\,\delta(t) - \dot{\delta}(t) + \ddot{\delta}(t). \end{aligned} \tag{58}$$

Inverse Laplace transform using partial fractions (cont.)

We see that the inverse Laplace transform of the polynomial part consists of impulses and their derivatives.
Now suppose that $R(s)$ is strictly proper. Writing $N(s)$ and $D(s)$ in terms of their factors we can write

$$R(s) = \frac{K\,(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \tag{59}$$

where z_i, p_j are in general complex and possibly repeated and denote the zeros and poles of $R(s)$ and $m < n$ since $R(s)$ is assumed to be strictly proper.
We consider the partial fraction expansion of $R(s)$ for several different cases.

Poles are distinct ($p_i \neq p_j$, $i \neq j$)

In this case the partial fraction expansion is

$$R(s) = \frac{A_1}{s - p_1} + \cdots + \frac{A_n}{s - p_n} \tag{60}$$

and

$$\begin{aligned} A_i &= R(s)\ (s - p_i)\big|_{s=p_i} \\ &= \frac{K\,(p_i - z_1)(p_i - z_2)\cdots(p_i - z_m)}{(p_i - p_1)(p_i - p_2)\cdots(p_i - p_{i-1})(p_i - p_{i+1})\cdots(p_i - p_n)} \end{aligned} \tag{61}$$

Once the A_i's are obtained from (61), we have

$$\mathcal{L}_{-}^{-1}\left\{R(s)\right\} = A_1\, e^{p_1\, t} + A_2\, e^{p_2\, t} + \cdots + A_n\, e^{p_n\, t}. \tag{62}$$

Note that in (62) the exponents correspond to the poles of $R(s)$. The A_i are called the **residues** associated with the poles P_i respectively.

Poles are distinct ($p_i \neq p_j$, $i \neq j$) (cont.)

Example

$$R(s) = \frac{s-1}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2} \tag{63}$$

where

$$A_1 = \left.\frac{(s-1)\cancel{(s+1)}}{\cancel{(s+1)}(s+2)}\right|_{s=-1} = -2, \tag{64}$$

$$A_2 = \left.\frac{(s-1)\cancel{(s+2)}}{(s+1)\cancel{(s+2)}}\right|_{s=-2} = 3. \tag{65}$$

Thus

$$\mathcal{L}_{-}^{-1}\left\{R(s)\right\} = -2\,e^{-t} + 3\,e^{-2\,t}. \tag{66}$$

Repeated Poles

a) **If one pole is repeated**, the partial fraction formula (60) has to be modified. Suppose that

$$R(s) = \frac{N(s)}{(s-p)^k} \tag{67}$$

where degree $N(s) < k$. Then it can be shown that the partial fraction expansion of $R(s)$ has the form:

$$R(s) = \frac{A_1}{s-p} + \frac{A_2}{(s-p)^2} + \cdots + \frac{A_k}{(s-p)^k}. \tag{68}$$

Multiplying by $(s-p)^k$ on both sides in (68) we have

$$R(s)\,(s-p)^k = N(s) = A_k + A_{k-1}(s-p) + \cdots + A_1(s-p)^{k-1}. \tag{69}$$

We have A_k by substituting $s = p$ in (69) since the rest terms become zeros.

$$A_k = N(s)|_{s=p}. \tag{70}$$

Repeated Poles (cont.)

Next, we differentiate $N(s)$ once and then we have

$$\frac{d}{d\,s}N(s) = A_{k-1} + 2\,A_{k-2}(s-p) + \cdots + (k-1)A_1(s-p)^{k-2}. \quad (71)$$

By substituting $s = p$ once again, we have

$$A_{k-1} = \left.\frac{d}{d\,s}N(s)\right|_{s=p}. \quad (72)$$

Thus we obtain A_{k-j} for $j = 0, \ldots, k-1$ by differentiating $N(s)$ j times and substituting $s = p$. This series of computations is

Repeated Poles (cont.)

summarized as follows.

$$N(s)\big|_{s=p} = A_k \tag{73}$$

$$\frac{d}{d\,s}N(s)\bigg|_{s=p} = A_{k-1} \tag{74}$$

$$\frac{1}{2}\frac{d^2}{d\,s^2}N(s)\bigg|_{s=p} = A_{k-2} \tag{75}$$

$$\frac{1}{3\cdot 2}\frac{d^3}{d\,s^3}N(s)\bigg|_{s=p} = A_{k-3} \tag{76}$$

$$\vdots \tag{77}$$

$$\frac{1}{j\,!}\frac{d^j}{d\,s^j}N(s)\bigg|_{s=p} = A_{k-j}, \text{ for } j = 0, \ldots, k-1. \tag{78}$$

Repeated Poles (cont.)

b) **If two or more poles are repeated**, we show that the treatment above also can be applied to find the numerator constants. Suppose

$$R(s) = \frac{N(s)}{(s-p)^k (s-q)^l}. \tag{79}$$

Then the partial fraction expansion gives

$$\begin{aligned} R(s) &= \frac{A_1}{s-p} + \frac{A_2}{(s-p)^2} + \cdots + \frac{A_k}{(s-p)^k} \\ &+ \frac{B_1}{s-q} + \frac{B_2}{(s-q)^2} + \cdots + \frac{B_l}{(s-q)^l}. \end{aligned} \tag{80}$$

Introduce

$$R_A(s) := \frac{R(s)}{(s-q)^l}, \tag{81}$$

$$R_B(s) := \frac{R(s)}{(s-p)^k}. \tag{82}$$

Repeated Poles (cont.)

In order to find B_l, we multiply $(s-q)^l$ on both sides of (80).

$$\begin{aligned} R(s)(s-q)^l &= \frac{N(s)}{(s-p)^k} = R_B(s) \\ &= (s-q)^l \underbrace{\left(\frac{A_1}{s-p} + \cdots + \frac{A_k}{(s-p)^k}\right)}_{=:A(s)} \\ &\quad + B_l + B_{l-1}(s-q) + \cdots + B_1(s-q)^{l-1}. \end{aligned} \tag{83}$$

Then,

$$\begin{aligned} B_l &= \left[R_B(s) - \left((s-q)^l A(s) + B_{l-1}(s-q) + \cdots + B_1(s-q)^{l-1}\right)\right]\Big|_{s=q} \\ &= R_B(s)|_{s=q} - \left((s-q)^l A(s) + B_{l-1}(s-q) + \cdots + B_1(s-q)^{l-1}\right)\Big|_{s=q} \\ &= R_B(s)|_{s=q}\,. \end{aligned} \tag{84}$$

Repeated Poles (cont.)

For B_{l-1}, we differentiate $R_B(s)$ one time.

$$\begin{aligned} \frac{d}{d\,s} R_B(s) = l(s-q)^{l-1} A(s) + (s-q)^l \left(\frac{d}{d\,s} A(s) \right) \\ + B_{l-1} + \cdots + (l-1) B_1 (s-q)^{l-2}. \end{aligned} \quad (85)$$

Thus, when we substitute $s = q$ in (85) we have

$$B_{l-1} = \left. \frac{d}{d\,s} R_B(s) \right|_{s=q}, \quad (86)$$

as the rest of the terms in (85) become zeros. In particular, for $(s-q)^l A(s)$, the multiplication by $(s-q)^l$ preserves the $(s-q)$

Repeated Poles (cont.)

factor even after $l-1$ times of differentiation of $R_B(s)$. This leads us to the following general form:

$$\left.\frac{1}{j\,!}\frac{d^j}{d\,s^j}R_A(s)\right|_{s=p} = A_{k-j},\ \text{for } j = 0,\ \ldots,\ k-1, \quad (87)$$

$$\left.\frac{1}{r\,!}\frac{d^r}{d\,s^r}R_B(s)\right|_{s=q} = B_{l-r},\ \text{for } r = 0,\ \ldots,\ l-1, \quad (88)$$

Repeated Poles (cont.)

Example

Suppose

$$R(s) = \frac{s-1}{(s+1)^3(s+2)} \tag{89}$$

then the partial fraction expansion of $R(s)$ is of the form

$$R(s) = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3} + \frac{B}{s+2}. \tag{90}$$

Clearly

$$\begin{aligned} B &= R(s)\,(s+2)\big|_{s=-2} \\ &= \left.\frac{(s-1)\cancel{(s+2)}}{(s+1)^3\cancel{(s+2)}}\right|_{s=-2} = 3 \end{aligned} \tag{91}$$

Repeated Poles (cont.)

Example (cont.)

and

$$A_3 = R(s)\,(s+1)^3\Big|_{s=-1} = \left.\frac{(s-1)\cancel{(s+1)^3}}{\cancel{(s+1)^3}(s+2)}\right|_{s=-1} = -2. \tag{92}$$

Since (90) must hold for "almost all" values of s, we set $s = 0$ and $s = 1$ to get

$$R(0) = -\frac{1}{2} = A_1 + A_2 - 2 + \frac{3}{2} \tag{93}$$

$$R(1) = 0 = \frac{A_1}{2} + \frac{A_2}{4} - \frac{2}{8} + \frac{3}{3}. \tag{94}$$

Solving for A_1 and A_2 we get

$$A_1 = -3, \quad A_2 = 3. \tag{95}$$

Repeated Poles (cont.)

Example (cont.)

Therefore

$$\mathcal{L}_{-}^{-1}\left\{R(s)\right\} = -3\,e^{-t} + 3\,t\,e^{-t} - \frac{2}{2!}\,t^{2}\,e^{-t} + 3\,e^{-2\,t}. \qquad (96)$$

Repeated Poles (cont.)

Example

Suppose

$$R(s) = \frac{s-1}{(s+1)^3(s+2)^2} \tag{97}$$

then the partial fraction expansion of $R(s)$ is of the form

$$R(s) = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3} + \frac{B_1}{s+2} + \frac{B_2}{(s+2)^2}. \tag{98}$$

Repeated Poles (cont.)

Example (cont.)

So

$$R_B(s) = \frac{s-1}{(s+1)^3}. \tag{99}$$

$$B_2 = \left.\frac{s-1}{(s+1)^3}\right|_{s=-2} = 3. \tag{100}$$

$$\begin{aligned} B_1 &= \left.\frac{d}{d\,s}\frac{s-1}{(s+1)^3}\right|_{s=-2} \\ &= \left.\frac{(s+1)^3 - 3(s-1)(s+1)^2}{(s+1)^6}\right|_{s=-2} \\ &= 8. \end{aligned} \tag{101}$$

Complex poles

If $R(s)$ has a pair of complex conjugate poles we could use (60) and get complex values for A_i. An alternative approach using real values is as follows. Write the second order factor with complex roots as

$$(s + \sigma)^2 + \omega^2 \tag{102}$$

and suppose

$$R(s) = \frac{N(s)}{(s + \sigma)^2 + \omega^2} \tag{103}$$

with degree $N(s) < 2$. Then the partial fraction expansion of (103) can be written without loss of generality as:

$$R(s) = \frac{A\,(s + \sigma)}{(s + \sigma)^2 + \omega^2} + \frac{B\,\omega}{(s + \sigma)^2 + \omega^2}. \tag{104}$$

The coefficients A and B can be found by equating coefficients in (104). Once A and B are known

$$\mathcal{L}_{-}^{-1}\left\{R(s)\right\} = A\,e^{-\sigma\,t}\,cos\,\omega\,t + B\,e^{-\sigma\,t}\,sin\,\omega\,t. \tag{105}$$

Complex poles (cont.)

Example

Suppose

$$R(s) = \frac{s-1}{(s^2+3\,s+4)}. \tag{106}$$

Write

$$\begin{aligned} s^2+3\,s+4 &= (s+\sigma)^2+\omega^2 \\ &= s^2+2\,\sigma\,s+\sigma^2+\omega^2 \end{aligned} \tag{107}$$

so that

$$\sigma = \frac{3}{2}, \quad \omega = \frac{\sqrt{7}}{2}. \tag{108}$$

Complex poles (cont.)

Example (cont.)

Now from (104)

$$R(s) = \frac{A\,s + A\,\sigma + B\,\omega}{(s^2 + 3\,s + 4)} = \frac{s - 1}{s^2 + 3s + 4} \tag{109}$$

so that $A = 1$ and

$$\frac{3 \cdot 1}{2} + \frac{B\,\sqrt{7}}{2} = -1 \tag{110}$$

giving

$$B = \frac{-5}{\sqrt{7}}. \tag{111}$$

Therefore

$$\mathcal{L}_{-}^{-1}\left\{R(s)\right\} = e^{-\frac{3}{2}\,t} cos\,\frac{\sqrt{7}}{2}\,t - \frac{5}{\sqrt{7}}\,sin\,\frac{\sqrt{7}}{2}\,t. \tag{112}$$

Forced Response of LTI Systems to Exponential Inputs

Consider a Linear Time Invariant system described by the differential equation connecting input $u(\cdot)$ to output $y(\cdot)$:

$$\begin{aligned} a_n \frac{d^n y}{d t^n} + a_{n-1} \frac{d^{n-1} y}{d t^{n-1}} + \cdots a_1 \frac{d y}{d t} + a_0 y(t) \\ = b_m \frac{d^m u}{d t^m} + b_{m-1} \frac{d^{m-1} u}{d t^{m-1}} + \cdots b \frac{d u}{d t} + b_0 u(t). \end{aligned} \tag{113}$$

Write the operator

$$D^k := \frac{d^k}{d t^k}, \quad k = 1, 2, \ldots \tag{114}$$

so that (113) can be rewritten as

$$a(D) y(t) = b(D) u(t). \tag{115}$$

Forced Response of LTI Systems to Exponential Inputs (cont.)

Now suppose that the input is the exponential signal

$$u(t) = e^{s^* t}. \tag{116}$$

The $e^{s^* t}$ component of the forced response must be of the form

$$y(t) = C\, e^{s^* t}. \tag{117}$$

Substituting (117) in (113) or (115) we see that

$$a(s^*)\, C\, e^{s^* t} = b(s^*)\, e^{s^* t} \tag{118}$$

so that, assuming $a(s^*) \neq 0$,

$$C = \frac{b(s^*)}{a(s^*)} = H(s^*) \tag{119}$$

Forced Response of LTI Systems to Exponential Inputs (cont.)

where

$$H(s) = \frac{b(s)}{a(s)} \tag{120}$$

denotes the system transfer function. The above analysis shows that the LTI system simply multiplies $e^{s^* t}$ by the gain $H(s^*)$ to generate the output $y(t) = H(s^*)\, e^{s^*\, t}$. Note that s^* may be real, imaginary or complex and this makes the above result very useful. Note also that the forced response will also contain exponential terms corresponding to the system poles, that is the zeros of $a(s) = 0$. If the poles of $H(s)$ are in the open LHP these components decay exponentially as $t \to \infty$.

Forced Response of LTI Systems to Exponential Inputs (cont.)

Example

Consider an LTI system with transfer function

$$H(s) = \frac{s+1}{s^2+s+2}. \tag{121}$$

Find the forced response of the system to the inputs:
$a)\ e^t U(t), \quad b)\ \cos t\, U(t), \quad c)\ e^t \cos 2t\, U(t).$

Forced Response of LTI Systems to Exponential Inputs (cont.)

Example (cont.)

Solution. In a) $s^* = 1$, and the response is

$$y(t) = H(1)\, e^t = \frac{1}{2}\, e^t. \tag{122}$$

For b) write

$$\cos t = \frac{1}{2}\, e^{jt} + \frac{1}{2}\, e^{-jt} \tag{123}$$

and

$$y(t) = \frac{1}{2}\, H(j)\, e^{jt} + \frac{1}{2}\, H(-j)\, e^{-jt}. \tag{124}$$

Forced Response of LTI Systems to Exponential Inputs (cont.)

Example (cont.)

In c)

$$e^t \cos 2t = e^t \frac{e^{2jt} + e^{-2jt}}{2} = \frac{1}{2} e^{(1+2j)t} + \frac{1}{2} e^{(1-2j)t} \qquad (125)$$

so that

$$y(t) = \frac{1}{2} H(1+j2) e^{(1+2j)t} + \frac{1}{2} H(1-j2) e^{(1-2j)t}. \qquad (126)$$

Since the poles of $H(s)$ have negative real parts, the above responses are also the "steady state" responses, that is those that remain as $t \to \infty$.

Forced Response to Sinusoidal Inputs

Consider the input

$$u(t) = \cos \omega\, t \tag{127}$$

applied to an LTI system with transfer function $H(s)$ with real coefficients in the numerator and denominator. Since

$$\cos \omega\, t = \frac{1}{2}\, e^{j\,\omega\, t} + \frac{1}{2}\, e^{-j\,\omega\, t} \tag{128}$$

the forced response in the steady state is

$$y(t) = \frac{1}{2}\, H(j\,\omega)\, e^{j\,\omega\, t} + \frac{1}{2}\, H(-j\,\omega)\, e^{j\,\omega\, t}. \tag{129}$$

Since $H(s)$ has real coefficients

$$H(-j\,\omega) = H^*(j\,\omega) \tag{130}$$

Forced Response to Sinusoidal Inputs (cont.)

and so (129) reduces to

$$y(t) = |H(j\,\omega)| \cos\left(\omega\, t + \underline{/H(j\,\omega)}\right) \tag{131}$$

which shows that the LTI system responds to $\cos \omega\, t$ by putting out the cosine wave of the same frequency **amplified** by $|H(j\,\omega)|$ and **phase-shifted** by $\underline{/H(j\,\omega)}$. It can be shown similarly that the response to

$$u(t) = \sin \omega\, t \tag{132}$$

is

$$y(t) = |H(j\,\omega)| \sin\left(\omega\, t + \underline{/H(j\,\omega)}\right). \tag{133}$$

Forced Response to Sinusoidal Inputs (cont.)

Example

Find the forced response of the system with transfer function $\dfrac{s+1}{s^2}$, to the input $2\cos t$.

Solution. The sinusoidal component of the output, from (131), is given by,

$$\begin{aligned} y(t) &= 2\left|\frac{1+j}{j^2}\right| \cos\left(\omega\, t + \angle \tfrac{1+j}{j^2}\right) \\ &= 2\sqrt{2}\cos\left(t - \frac{3\,\pi}{4}\right). \end{aligned} \tag{134}$$

Note that the steady state response of the system will also contain steps and ramps due to the s^2 term in the denominator of the transfer function.

Forced Response to Sinusoidal Inputs (cont.)

Example

Consider the system,

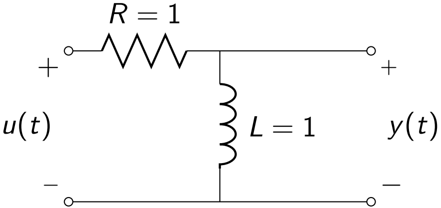

Figure:

Forced Response to Sinusoidal Inputs (cont.)

Example (cont.)

The inductor voltage can be written as follows.

$$L\left(\frac{\dot{u}-\dot{y}}{R}\right)=y, \tag{135}$$

$$\dot{y}+\frac{R}{L}y=\dot{u}. \tag{136}$$

The Laplace transform gives

$$Y(s)=\left(\frac{s}{s+\frac{R}{L}}\right)U(s). \tag{137}$$

Thus the transfer function is

$$H(s)=\frac{s}{s+1}. \tag{138}$$

Forced Response to Sinusoidal Inputs (cont.)

Example (cont.)

Now, we can obtain the forced response components of $y(t)$ corresponding to the following input signals.

a) $u(t) = e^t$,

$$y(t) = H(1)\, e^t = \frac{1}{2}\, e^t. \tag{139}$$

b) $u(t) = \cos 2t$,

$$\begin{aligned} y(t) &= |H(j2)| \cos\left(2t + \angle H(j2)\right) \\ &= \frac{4}{5} \cos\left(2t + tan^{-1}(0.5)\right). \end{aligned} \tag{140}$$

Forced Response to Sinusoidal Inputs (cont.)

Example (cont.)

c) $u(t) = \sin t$

$$\begin{aligned} y(t) &= |H(j)| \sin\left(t + \angle H(j)\right) \\ &= \frac{1}{2}\cos\left(t + \frac{\pi}{4}\right). \end{aligned} \tag{141}$$

d) $u(t) = e^t \cos(2t) = \dfrac{e^{t+j\,2\,t}}{2} + \dfrac{e^{t-j\,2\,t}}{2}$

$$\begin{aligned} y(t) &= H(1+j2)\frac{e^{t+j\,2\,t}}{2} + H(1-j2)\frac{e^{t-j\,2\,t}}{2} \\ &= \frac{1}{2}|H(1+j2)|\left(e^{t+2\,j\,t+j\angle H(1+j2)}\right) \\ &+ \frac{1}{2}|H(1-j2)|\left(e^{t-2\,j\,t+j\angle H(1-j2)}\right). \end{aligned} \tag{142}$$

Since the system transfer function has stable poles, (142) is also the steady state response.

Exercises

Exercise 1

If $U(t)$ denotes the unit step, show that

$$\frac{d\,U(t)}{dt} = \delta(t). \tag{143}$$

Exercises

Exercise 2

Solve using Laplace transforms and identifying the transfer function, the zero input and zero state responses. ($U(t)$: unit step)

a) $\dot{y}(t) + y(t) = \dot{u}(t) - u(t)$, $u(t) = U(t)$, $y(0^-) = 1$

b) $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{u}(t) + u(t)$, $u(t) = U(t)$, $y(0^-) = 1$, $\dot{y}(0^-) = -1$

c) $\ddot{y}(t) + 5\dot{y}(t) + 7y(t) = \dot{u}(t) - u(t)$, $u(t) = U(t)$, $y(0^-) = 1$, $\dot{y}(0^-) = 1$

d) $\dddot{y}(t) + 3\ddot{y}(t) + 5\dot{y}(t) + 3y(t) = \dot{u}(t) - u(t)$, $u(t) = U(t)$, $y(0^-) = 1$, $\dot{y}(0^-) = -1$

Exercises

Exercise 3

Evaluate the following functions. ($*$ denotes convolution)

a) $g(t) = e^t \, * \, e^{-t}$

b) $g(t) = U(t) \, * \, e^t$

c) $g(t) = t \, * \, sin\, t$

d) $g(t) = \delta(t) \, * \, t^2$

e) $g(t) = sin\, t \, * \, cos\, t$