ECEN 605

LINEAR SYSTEMS

Lecture 5

Mathematical Representation of Physical Systems I

A Conceptual Model

We regard a **system** as consisting of **inputs** u_1, u_2, ..., u_r, **outputs** denoted y_1, y_2, ..., y_m, and **internal** or **state** variables denoted by x_1, x_2, ..., x_n . Using vector notation

$$\mathbf{y} := \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \mathbf{u} := \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{1}$$

denote the m-output vector, r-input vector and n-state vector respectively. For example, in a circuit $\mathbf{y}$ and $\mathbf{u}$ might represent terminal voltages and input voltages and $\mathbf{x}$ the vector of internal currents and voltages across all branches. Considering human body, the inputs may be food intake, beverages consumed and medication administered, and the output $\mathbf{y}$ may be body temperature, blood pressure and pulse rate while the internal

A Conceptual Model (cont.)

variables **x** may be heart rate, blood sugar concentration, brain activity level and various hormone levels.

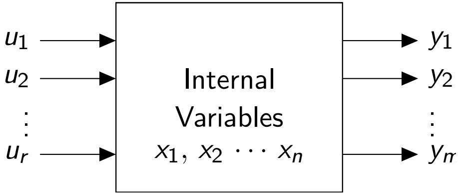

Figure: A conceptual model

Static Linear Models

A static model is one where the variables describing the system are related by algebraic equations and no derivative with respect to time is involved. When the algebraic equations describing the system are linear with constant coefficients we have a **static** linear time invariant (LTI) system.

For static LTI systems we propose a general model of the form:

$$\mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} = \mathbf{0} \tag{2a}$$

$$\mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} = \mathbf{y} \tag{2b}$$

where $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ are matrices of size $n \times n$, $n \times r$, $m \times n$ and $m \times r$ respectively. Let us illustrate this concept by examples.

Static Linear Models (cont.)

Example (DC circuit)

Consider the resistive circuit below with currents and voltages as assigned.

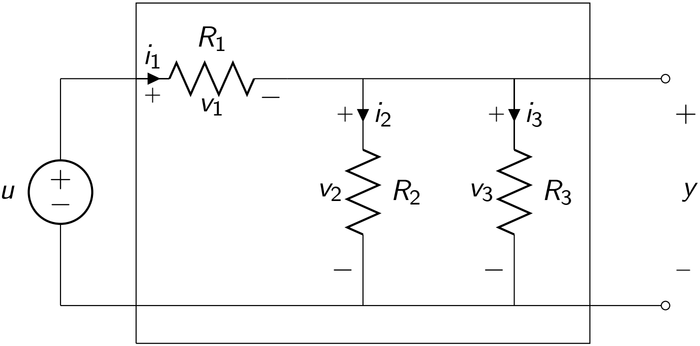

Figure: A dc circuit

Static Linear Models (cont.)

Example (DC circuit (cont.))

The input u is the voltage source, the output voltage is y and the internal variables are the branch currents i_j, the branch voltages v_j across the resistors R_j, $j = 1, 2, 3$. Writing Kirchhoff's equations for the circuit, namely

$$\text{1) algebraic sum of currents entering a node} = 0 \tag{3a}$$

$$\text{2) algebraic sum of voltage drops around a loop} = 0 \tag{3b}$$

we have

$$i_1 = i_2 + i_3 \tag{2.3a'}$$

and

$$\left.\begin{aligned} u &= v_1 + v_2 \\ v_2 &= v_3 \end{aligned}\right\} \tag{2.3b'}$$

Static Linear Models (cont.)

Example (DC circuit (cont.))

and by Ohm's law, we have

$$v_j = R_j i_j, \quad j = 1, 2, 3. \tag{4}$$

Note that (2.3a') and (2.3b') are structural equations depending on the node and loop structure of the system and contains no parameters, whereas only (4) contains the parameters R_j, $j = 1, 2, 3$. Rewriting (2.3a'), (2.3b') and (4) in matrix form, we have

Static Linear Models (cont.)

Example (DC circuit (cont.))

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -R_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -R_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -R_3 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{B}=\mathbf{b}} u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5a}$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}=\mathbf{c}} \mathbf{x} + \underbrace{0}_{\mathbf{D}=d} u = y. \tag{5b}$$

which is of the form (2), with $m = 1$ (single output), $r = 1$ (single input) and $n = 6$. Accordingly $\mathbf{C}$, $\mathbf{B}$ and $\mathbf{D}$ are of size 1×6, 6×1 and 1×1 respectively and $\mathbf{A}$ is 6×6.

Static Linear Models (cont.)

Example (DC circuit (cont.))

In (5) the mapping from input u to output y is made through the intermediate internal variables $\mathbf{x}$. A direct relationship between u and y can be obtained by eliminating $\mathbf{x}$, as shown below. This can be done systematically by using Cramer's Rule.
Writing (5) using

$$\mathbf{z} := \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} \tag{6}$$

so that

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{c} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} = \begin{bmatrix} -\mathbf{b} \\ d \end{bmatrix} u. \tag{7}$$

Static Linear Models (cont.)

Example (DC circuit (cont.))

Applying Cramer's Rule to (7) and solving for y, we have

$$y = \frac{\left|\begin{bmatrix} \mathbf{A} & -\mathbf{b} \\ -\mathbf{c} & \mathbf{d} \end{bmatrix}\right|}{\left|\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{c} & 1 \end{bmatrix}\right|} u \tag{8a}$$

$$= \underbrace{\frac{\left|\begin{bmatrix} \mathbf{A} & -\mathbf{b} \\ -\mathbf{c} & \mathbf{d} \end{bmatrix}\right|}{|\mathbf{A}|}}_{\mathbf{G}} u. \tag{8b}$$

Equation (8) shows that the "gain" $\mathbf{G}$ of the system $y = \mathbf{G}\, u$ is given by

$$\mathbf{G} = \frac{|\mathbf{T}|}{|\mathbf{A}|}, \quad \mathbf{T} := \begin{bmatrix} \mathbf{A} & -\mathbf{b} \\ -\mathbf{c} & d \end{bmatrix}. \tag{9}$$

Static Linear Models (cont.)

Example (DC circuit (cont.))

So

$$\mathbf{G} = \frac{\left|\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ -R_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -R_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -R_3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}\right|}{\left|\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -R_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -R_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -R_3 & 0 & 0 & 1 \end{bmatrix}\right|} \tag{10}$$

$$= \frac{-R_2\,R_3}{-R_1\,R_2 - R_1\,R_3 - R_2\,R_3}.$$

Static Linear Models (cont.)

Example (2 input 2 output DC circuit)

Consider the circuit below with inputs u_1 (voltage source) u_2 (current source) and outputs y_1 and y_2:

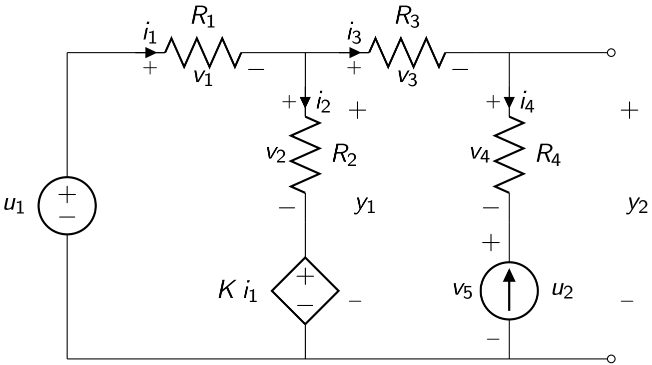

Figure: A dc circuit

Static Linear Models (cont.)

Example (2 input 2 output DC circuit (cont.))

The system equations are:

$$
\begin{aligned}
i_1 &= i_2 + i_3 \\
i_3 &= i_4 \\
i_4 &= -u_2 \\
u_1 &= v_1 + v_2 + K\, i_1 \\
K\, i_1 &= v_2 + v_3 + v_5 + v_4 \\
v_j &= R_j\, i_j, \quad j = 1,\, 2,\, 3,\, 4 \\
y_1 &= v_2 + K\, i_1 \\
y_2 &= v_4 + v_5.
\end{aligned}
\tag{11}
$$

Static Linear Models (cont.)

Example (2 input 2 output DC circuit (cont.))

In matrix notation with

$$\mathbf{y} := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \mathbf{u} := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} i_1 \\ \vdots \\ i_4 \\ v_1 \\ \vdots \\ v_5 \end{bmatrix} \tag{12}$$

we may rewrite (11) as:

$$\begin{aligned} \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} &= \mathbf{0} \\ \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} &= \mathbf{y} \end{aligned} \tag{13}$$

Static Linear Models (cont.)

Example (2 input 2 output DC circuit (cont.))

where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -K & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -R_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -R_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -R_3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 1 & 0 \\ K & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix} \tag{14}$$

Static Linear Models (cont.)

Example (2 input 2 output DC circuit (cont.))

$$\mathbf{B} = \begin{bmatrix}\mathbf{b_1} & \mathbf{b_2}\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix}\mathbf{c_1} \\ \mathbf{c_2}\end{bmatrix} = \begin{bmatrix} K & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{15}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Static Linear Models (cont.)

Example (2 input 2 output DC circuit (cont.))

In the general model (2) let y_i denote the i^{th} output and u_j the j^{th} input. If $\mathbf{c}_i$ denotes the i^{th} row of $\mathbf{C}$, $\mathbf{b}_j$ the j^{th} column of $\mathbf{B}$ and d_{ij} the (i, j) element of $\mathbf{D}$, it follows from (2) that

$$y_i = \frac{|\mathbf{T}_{i1}|}{|\mathbf{A}|}\, u_1 + \frac{|\mathbf{T}_{i2}|}{|\mathbf{A}|}\, u_2 + \cdots \frac{|\mathbf{T}_{ir}|}{|\mathbf{A}|}\, u_r \tag{16}$$

where

$$\mathbf{T}_{ij} = \begin{bmatrix} \mathbf{A} & -\mathbf{b}_j \\ -\mathbf{c}_i & d_{ij} \end{bmatrix}, \quad \substack{i=1,2,\ldots,m,\\ j=1,2,\ldots,r.} \tag{17}$$

Static Linear Models (cont.)

Example (2 input 2 output DC circuit (cont.))

For the 2 input 2 output example

$$\begin{aligned} y_1 &= \frac{|\mathbf{T}_{11}|}{|\mathbf{A}|}\, u_1 + \frac{|\mathbf{T}_{12}|}{|\mathbf{A}|}\, u_2, \\ y_2 &= \frac{|\mathbf{T}_{21}|}{|\mathbf{A}|}\, u_1 + \frac{|\mathbf{T}_{22}|}{|\mathbf{A}|}\, u_2. \end{aligned} \tag{18}$$

The above equation can be written in terms of the "gains" $\mathbf{G}_{ij}$:

$$\mathbf{G}_{ij} = \frac{|\mathbf{T}_{ij}|}{|\mathbf{A}|}, \quad \substack{i=1,2,\ldots,m, \\ j=1,2,\ldots,r,} \tag{19}$$

Static Linear Models (cont.)

Example (2 input 2 output DC circuit (cont.))

so that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix}}_{\mathbf{G}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{20}$$

or

$$\mathbf{y} = \mathbf{G}\,\mathbf{u} \tag{21}$$

and G represents the gain matrix.

Static Linear Models (cont.)

Example (2 input 2 output DC circuit (cont.))

$$
\mathbf{G}_{11} = \frac{\left|\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -K & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ -R_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 1 & 0 & 0 \\ K & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ K & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}\right|}{\left|\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -K & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -R_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -R_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -R_3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 1 & 0 \\ K & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}\right|} = \frac{K + R_2}{K + R_1 + R_2}. \tag{22}
$$

Static Linear Models (cont.)

Example (2 input 2 output DC circuit (cont.))

Similarly,

$$\mathbf{G}_{12} = \frac{R_1\,R_2}{K + R_1 + R_2}, \tag{23}$$

$$\mathbf{G}_{21} = \frac{K - R_2}{K + R_1 + R_2}, \tag{24}$$

$$\mathbf{G}_{22} = \frac{-2\,K\,R_2 + K\,R_3 - R_1\,R_2 + R_1\,R_3 + R_2\,R_3}{K + R_1 + R_2}. \tag{25}$$

Static Linear Models (cont.)

Remark

Although we have only dealt with circuits here, the general model (2) can be developed for static mass spring systems, hydraulic networks in st steady state flow and truss structures. These are discussed in the recent monograph "Linear Systems: A Measurement Based Approach" by D. N. Mohsenizadeh, L. H. Keel and S. P. Bhattacharyya, Springer, 2013.