ECEN 605

LINEAR SYSTEMS

Lecture 11

Structure of LTI Systems III
– Realization Theory

Single Input or Single Output Systems
Scalar Systems

Let

$$g(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{1}$$

where a_i, b_j are real and n is the order of the system. One possible realization for this system is shown in Figure 1:

Single Input or Single Output Systems (cont.) Scalar Systems

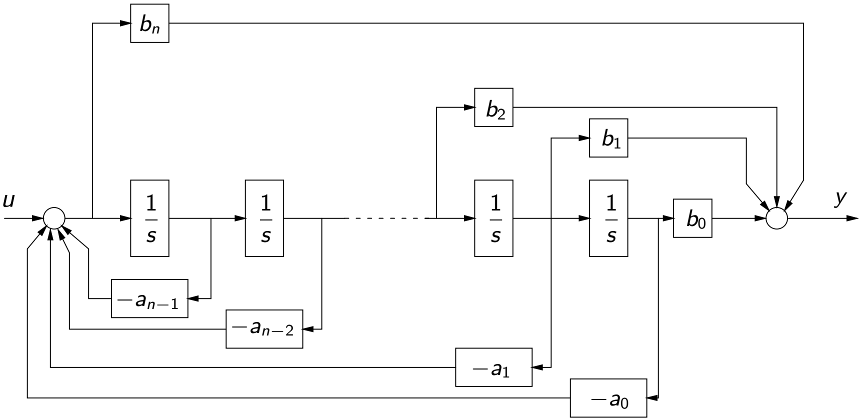

Figure 1: A Typical Realization

Single Input or Single Output Systems (cont.)
Scalar Systems

Another possibility is to "divide out" the eq. (1) and rewrite as follows:

$$g(s) = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d. \qquad (2)$$

This leads to the following realization (see Figure 2):

Single Input or Single Output Systems (cont.) Scalar Systems

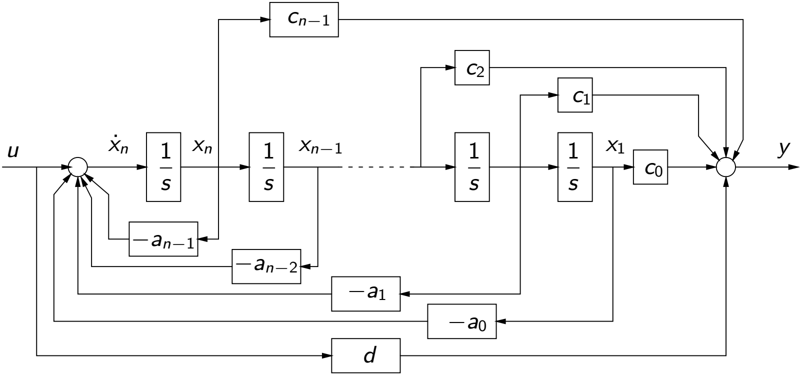

Figure 2: A Typical Realization

Single Input or Single Output Systems (cont.) Scalar Systems

Assigning state variables as in Figure 2, we have the set of equations:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
&\vdots \\
&\vdots \\
\dot{x}_{n-1}(t) &= x_n(t) \\
\dot{x}_n(t) &= -a_0 x_1(t) - a_1 x_2(t) - \cdots - a_{n-1} x_n(t) + u(t) \\
\\
y(t) &= c_0 x_1(t) + c_1 x_2(t) + \cdots + c_{n-1} x_n(t) + du(t).
\end{aligned} \tag{3}$$

Single Input or Single Output Systems (cont.) Scalar Systems

Therefore, in this realization, we write

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C(t)x + Du(t) \end{aligned}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} := A_c, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} := b_c \\ C &= \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \end{bmatrix} := c_c, \qquad D = d. \end{aligned} \tag{4}$$

Single Input or Single Output Systems (cont.)
Scalar Systems

A compact notation for this is the so-called *packed matrix* notation:

$$g(s) = \left[\begin{array}{cc} A_c & b_c \\ & \\ c_c & d \end{array} \right] \tag{5}$$

The special form of (A_c, b_c) is called the *controllable companion* form (or controllable cannonical form). Clearly, such form is guaranteed to be controllable.

Single Input or Single Output Systems (cont.) Scalar Systems

Laplace transformation of (3) assuming $x_i(0) = 0$ is,

$$\begin{aligned} u(s) &= sX_n(s) + a_{n-1}X_n(s) + a_{n-2}X_{n-1} + \cdots + a_1X_2(s) + a_0X_1(s) \\ &= s^nX_1(s) + a_{n-1}s^{n-1}X_1(s) + a_{n-2}s^{n-2}X_1(s) + \cdots + a_1sX_1(s) + a_0X_1(s) \\ &= (s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1s + a_0)X_1(s) \end{aligned}$$

and

$$\begin{aligned} y(s) &= c_0X_1(s) + c_1X_2(s) + \cdots + c_{n-1}X_n(s) + du(s) \\ &= (c_0 + c_1s + \cdots + c_{n-1}s^{n-1})X_1(s) + du(s). \end{aligned}$$

Therefore,

$$\begin{aligned} y(s) &= \left(\frac{c_0 + c_1s + \cdots + c_{n-1}s^{n-1}}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1s + a_0} + d\right)u(s) \\ &= g(s)u(s). \end{aligned}$$

Single Input or Single Output Systems (cont.) Scalar Systems

Equivalently,

$$g(s) = c_c(sI - A_c)^{-1}b_c + d.$$

Notice that

$$\begin{aligned}
g^T(s) = g(s) &= \left[c_c(sI - A_c)^{-1}b_c + d\right]^T \\
&= \left[c_c(sI - A_c)^{-1}b_c\right]^T + d^T \\
&= b_c^T\left[(sI - A_c)^{-1}\right]^T c_c^T + d \quad (d \text{ is a constant}) \\
&= b_c^T\left[(sI - A_c)^T\right]^{-1} c_c^T + d \\
&= b_c^T(sI - A_c^T)^{-1}c_c^T + d.
\end{aligned}$$

Single Input or Single Output Systems (cont.)
Scalar Systems

Therefore, by defining

$$c_o := b_c^T, \quad A_o := A_c^T, \quad b_o := c_c^T, \quad d_o = d,$$

we have another realization, namely

$$g(s) = \left[\begin{array}{c|c} A_o & b_o \\ \hline c_o & d_o \end{array} \right]. \tag{6}$$

Single Input or Single Output Systems (cont.) Scalar Systems

Here,

$$A_o = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \qquad b_o = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} \quad (7)$$

$$c_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \qquad d_o = d.$$

The pair (c_o, A_o) is said to be in the *observable companion* (or cannonical) form. The realizations in eq. (4) and eq. (7) are *duals* of each other. There is a circuit corresponding to the observable realization given in eq. (7).

Single Input or Single Output Systems (cont.) Scalar Systems

Other Realizations

It is possible to get many other realizations by appropriate decompositions of the transfer function. For instance, writing

$$g(s) = g_1(s)g_2(s)\cdots g_k(s)$$

leads to the realization

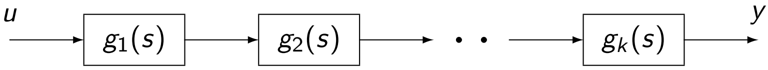

Figure 3: A Cascaded Realization

where each subsystem $g_i(s)$ should be proper or strictly proper and therefore realizable.

Single Input or Single Output Systems (cont.)
Scalar Systems

Likewise, writing

$$g(s) = h_1(s) + h_2(s) + \cdots + h_l(s)$$

where $h_i(s)$ is proper or strictly proper leads to the realization.

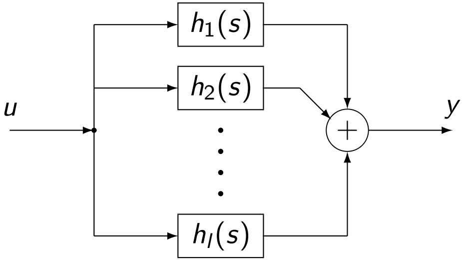

Figure 4: A Parallel Realization

In each case, state equations can be obtained by assigning a state variable as the output of each integrator.

Single Input or Single Output Systems
Multi-input Systems (MISO)

Consider a system with r inputs and 1 output:

$$G(s) = \underbrace{\begin{bmatrix} \frac{\bar{n}_1(s)}{d_1(s)} & \frac{\bar{n}_2(s)}{d_2(s)} & \cdots & \frac{\bar{n}_r(s)}{d_r(s)} \end{bmatrix}}_{\text{strictly proper part of } \bar{G}(s)} + \begin{bmatrix} d_1 & d_2 & \cdots & d_r \end{bmatrix}.$$

Let

$$d(s) = \mathrm{LCM}\,[d_1(s),\ d_2(s),\ \cdots,\ d_r(s)]$$

and rewrite the strictly proper part as

$$\bar{G}(s) = \frac{1}{d(s)} \begin{bmatrix} n_1(s) & n_2(s) & \cdots & n_r(s) \end{bmatrix}$$

where

$$\begin{aligned} d(s) &= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \\ n_i(s) &= c^i_{n-1}s^{n-1} + c^i_{n-2}s^{n-2} + \cdots + c^i_1 s + c^i_0, \qquad i = 1, 2, \cdots, r. \end{aligned}$$

Single Input or Single Output Systems (cont.) Multi-input Systems (MISO)

Then a possible realization is the observable companion form of order n:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & & & \vdots \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & -\,a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} c_0^1 & c_0^2 & \cdots & c_0^r \\ c_1^1 & c_1^2 & \cdots & c_1^r \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ c_{n-1}^1 & c_{n-1}^2 & \cdots & c_{n-1}^r \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} d_1 & d_2 & \cdots & d_r \end{bmatrix}.$$

Single Input or Single Output Systems
Multi-output Systems (SIMO)

If we have a single input multioutput system, we have

$$G(s) = \underbrace{\begin{bmatrix} G_1(s) \\ G_2(s) \\ \vdots \\ G_m(s) \end{bmatrix}}_{\text{strictly proper}} + \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}}_{D}.$$

We write as before

$$\begin{aligned} G_i(s) &= \frac{n_i(s)}{d(s)} \\ &= \frac{c^i_{n-1}s^{n-1} + c^i_{n-2}s^{n-2} + \cdots + c^i_1 s + c^i_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \qquad i = 1, 2, \cdots, m. \end{aligned}$$

Single Input or Single Output Systems (cont.) Multi-output Systems (SIMO)

Then a possible realization is the controllable companion form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_0^1 & c_1^1 & c_2^1 & \cdots & c_{n-1}^1 \\ c_0^2 & c_1^2 & c_2^2 & \cdots & c_{n-1}^2 \\ \vdots & & & & \vdots \\ c_0^m & c_1^m & c_2^m & \cdots & c_{n-1}^m \end{bmatrix}, \qquad D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}.$$

Multi-input Multi-output Systems (MIMO)

In the general case, write extract D at the constant

$$G(s) = \underbrace{\bar{G}(s)}_{\text{strictly proper}} + \underbrace{D}_{\text{constant}}$$

and proceed with the strictly proper part. We can either place all entries in each column over a common denominator or place each row over a common denominator.

Multi-input Multi-output Systems (MIMO) (cont.)

Doing columns, first we would have

$$\bar{G}(s) = \begin{bmatrix} \frac{n_{11}(s)}{d_1(s)} & \frac{n_{12}(s)}{d_2(s)} & \cdots & \frac{n_{1r}(s)}{d_r(s)} \\ \frac{n_{21}(s)}{d_1(s)} & \frac{n_{22}(s)}{d_2(s)} & \cdots & \frac{n_{2r}(s)}{d_r(s)} \\ \vdots & & & \vdots \\ \frac{n_{m1}(s)}{d_1(s)} & \frac{n_{m2}(s)}{d_2(s)} & \cdots & \frac{n_{mr}(s)}{d_r(s)} \end{bmatrix}$$

where

$$\begin{aligned} d_i(s) &= s^{n_i} + a^i_{n_i-1}s^{n_i-1} + \cdots + a^i_1 s + a^i_0 \\ n_{ki}(s) &= c^{ki}_{n_i-1}s^{n_i-1} + \cdots + c^{ki}_1 s + c^{ki}_0 \\ & \qquad k = 1, 2, \cdots, m; \quad i = 1, 2, \cdots, r. \end{aligned}$$

Multi-input Multi-output Systems (MIMO) (cont.)

A possible realization is

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_r \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mr} \end{bmatrix}, \qquad D = D$$

Multi-input Multi-output Systems (MIMO) (cont.)

where

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0^i & -a_1^i & -a_2^i & \cdots & -a_{n_i-1}^i \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$c_{ki} = \begin{bmatrix} c_0^{ki} & c_1^{ki} & \cdots & c_{n_i-1}^{ki} \end{bmatrix}, \quad k \in [1, 2, \cdots, m]; \quad i \in [1, 2, \cdots, r].$$

This form is controllable.

Multi-input Multi-output Systems (MIMO) (cont.)

The dual procedure with common denominators over rows is as follows:

$$\bar{G}(s) = \begin{bmatrix} \frac{n_{11}(s)}{d_1(s)} & \frac{n_{12}(s)}{d_1(s)} & \cdots & \frac{n_{1r}(s)}{d_1(s)} \\ \frac{n_{21}(s)}{d_2(s)} & \frac{n_{22}(s)}{d_2(s)} & \cdots & \frac{n_{2r}(s)}{d_2(s)} \\ \vdots & & & \vdots \\ \frac{n_{m1}(s)}{d_m(s)} & \frac{n_{m2}(s)}{d_m(s)} & \cdots & \frac{n_{mr}(s)}{d_m(s)} \end{bmatrix}$$

where

$$\begin{aligned} d_i(s) &= s^{n_i} + a^i_{n_i-1} s^{n_i-1} + \cdots + a^i_1 s + a^i_0 \\ n_{ij}(s) &= c^{ij}_{n_i-1} s^{n_i-1} + \cdots + c^{ij}_1 s + c^{ij}_0 . \end{aligned}$$

Multi-input Multi-output Systems (MIMO) (cont.)

Then the observable realization is

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mr} \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_m \end{bmatrix}, \qquad D = D$$

Multi-input Multi-output Systems (MIMO) (cont.)

where

$$A_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0^i \\ 1 & 0 & \cdots & 0 & -a_1^i \\ 0 & 1 & & & \vdots \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n_i-1}^i \end{bmatrix}, \qquad b_{ij} = \begin{bmatrix} c_0^{ij} \\ c_1^{ij} \\ \vdots \\ \vdots \\ c_{n_i-1}^{ij} \end{bmatrix}$$

$$c_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Likewise, this form is observable.