ECEN 605

LINEAR SYSTEMS

Lecture 12

Structure of LTI Systems IV
– Minimal Realizations

Controllability, Observability, and Minimality

Let $G(s)$ be a proper rational matrix and let $\{A, B, C, D\}$ be a realization. If A is $n \times n$, we say the order of the realization is n.

An important question is: Is it possible to realize $G(s)$ with a lower order dynamic system? If not, n is the *minimal order*. Otherwise, how do we find the minimal order?

This problem was completely solved by Kalman in a classical paper[1]. The solution involves the concepts of controllability and observability which are also important in other areas.

Controllability, Observability, and Minimality (cont.)

Theorem (Minimal Realization)

A realization $\{A, B, C, D\}$ of a proper rational matrix $G(s)$ is minimal iff (A, B) is controllable and (C, A) is observable.

This result is obtained by Kalman. It implies that if (A, B) is not controllable, the order can be reduced. Likewise, if (C, A) is not observable, the order can also be reduced.

[1] R.E. Kalman, "Irreducible Realizations and the Degree of a Rational Matrix," *SIAM J. Appl. Math.*, Vol. 13, pp. 520-544, June 1965

Coordinate Transformation and Order Reduction

If we set

$$x(t) = Tz(t) \qquad T \in \mathbb{R}^{n \times n}$$

where T is invertible, then we have

$$\begin{aligned} \dot{z}(t) &= T^{-1}ATz(t) + T^{-1}Bu(t) \\ y(t) &= CTz(t) + Du(t) \end{aligned}$$

as the new state equations in z.

Coordinate Transformation and Order Reduction (cont.)

It can be easily verified that the "new" transfer function is

$$\begin{aligned} CT(sI - T^{-1}AT)^{-1}T^{-1}B + D &= C(sI - A)^{-1}B + D \\ &= \text{old transfer function} \end{aligned}$$

and the new state space realization is related to the old one by relationship:

$$\{A, B, C, D\} \longrightarrow^{T} \{\underbrace{T^{-1}AT}_{A_{\text{new}}}, \underbrace{T^{-1}B}_{B_{\text{new}}}, \underbrace{CT}_{C_{\text{new}}}, \underbrace{D}_{D_{\text{new}}}\}.$$

This is called a *similarity transformation*.

Coordinate Transformation and Order Reduction (cont.)

The next two observations are crucial. If

$$T^{-1}AT = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad D = D,$$

we can see that

$$C(sI - A)^{-1}B + D = C_1(sI - A_1)^{-1}B_1 + D.$$

Similarly, if

$$T^{-1}AT = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CT = \begin{bmatrix} 0 & C_2 \end{bmatrix}, \quad D = D,$$

then

$$C(sI - A)^{-1}B + D = C_2(sI - A_2)^{-1}B_2 + D.$$

In the first case, the order is reduced from n to n_1 (size of A_1). In the second case, the order is reduced from n to n_2 (size of A_2).

Controllability Reduction

Let us regard

$$A \; : \; \mathcal{X} \longrightarrow \mathcal{X}$$

as a *linear operator*, and define

$$R := \left[\begin{array}{ccccc} B & AB & A^2B & \cdots & A^{n-1}B \end{array} \right],$$

the controllability matrix and let $\mathcal{R}$ denote the column span of R.

Controllability Reduction (cont.)

In other words if $\text{rank}[R] = n_1$, then $\mathcal{R}$ is the n_1 dimensional subspace spanned by the columns of R. Let $\{v_1, v_2, \cdots, v_{n_1}\}$ be a set of basis vectors for $\mathcal{R}$ and let $\{w_{n_1+1}, w_{n_1+2}, \cdots, w_n\}$ be $n - n_1$ vectors such that

$$T := \left[\begin{array}{cccc|cccc} v_1 & v_2 & \cdots & v_{n_1} & w_{n_1+1} & w_{n_1+2} & \cdots & w_n \end{array} \right]$$

is an $n \times n$ invertible matrix.

Controllability Reduction (cont.)

Lemma

$$T^{-1}AT = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \qquad T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{1}$$

where A_1 is $n_1 \times n_1$, B_1 is $n_1 \times r$. Then (A_1, B_1) is controllable. The pair of two matrices in (1) is called the Kalman controllable canonical form.

Controllability Reduction (cont.)

The proof of the lemma depends on the following fact.

Definition

Let $A \in \mathbb{R}^{n \times n}$. And, $\mathcal{V} \subset \mathbb{R}^n$ is a subspace. Then we say that $\mathcal{V}$ is A-invariant if $A\mathcal{V} \subset \mathcal{V}$, i.e., $v \in \mathcal{V}$ implies that $Av \in \mathcal{V}$.

Lemma

$\mathcal{R}$ is an A-invariant subspace and

$$\mathcal{B}(\text{column span of } B) \subset \mathcal{R}.$$

In fact $\mathcal{R}$ is the smallest A-invariant subspace containing $\mathcal{B}$.

Controllability Reduction (cont.)

Proof

Suppose $r \in \mathcal{R}$. Then

$$r = By_0 + ABy_1 + \cdots + A^{n-1}By_{n-1} \in \mathcal{R}$$

for some vectors $y_0, y_1, \cdots, y_{n-1}$. Then

$$Ar = ABy_0 + A^2By_1 + \cdots + A^nBy_{n-1}.$$

By the Cayley-Hamilton Theorem

$$A^n = \alpha_{n-1}A^{n-1} + \alpha_{n-2}A^{n-2} + \cdots + \alpha_1 A + \alpha_0 I.$$

Controllability Reduction (cont.)

Substituting this in the expression for Ar, we have

$$Ar = Bz_0 + ABz_1 + \cdots + A^{n-1}Bz_{n-1} \in \mathcal{R}$$

for some vectors z_0, z_1, $\cdots$, z_{n-1}. Therefore,

$$Ar \in \mathcal{R}.$$

Obviously, $\mathcal{B} \subset \mathcal{R}$.

Controllability Reduction (cont.)

To prove that $\mathcal{R}$ is the smallest such subspace, let $\mathcal{S}$ be a smaller subspace. Then

$$\mathcal{B} \subset \mathcal{S} \subset \mathcal{R}.$$

Applying A to both sides, we have

$$\begin{aligned}
&A\mathcal{B} \subset A\mathcal{S} \subset \mathcal{S} \subset \mathcal{R} \\
&A^2\mathcal{B} \subset A\mathcal{S} \subset \mathcal{S} \subset \mathcal{R} \\
&\vdots \\
&A^{n-1}\mathcal{B} \subset A\mathcal{S} \subset \mathcal{S} \subset \mathcal{R}.
\end{aligned}$$

Therefore,

$$\mathcal{R} := \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B} \subset \mathcal{S} \subset \mathcal{R}.$$

so that $\mathcal{S} = \mathcal{R}$.

Controllability Reduction (cont.)

Proof (Proof of the first lemma)

Eq. (1) is equivalent to the following

$$A \begin{bmatrix} v_1 & \cdots & v_{n_1} & \vdots & w_{n_1+1} & \cdots & w_n \end{bmatrix} =$$

$$\begin{bmatrix} v_1 & \cdots & v_{n_1} & \vdots & w_{n_1+1} & \cdots & w_n \end{bmatrix} \begin{bmatrix} A_1 & \vdots & A_3 \\ \cdots & & \cdots \\ A_4 & \vdots & A_2 \end{bmatrix},$$

$$\begin{bmatrix} b_1 & b_2 & \cdots & b_r \end{bmatrix} B = \begin{bmatrix} v_1 & \cdots & v_{n_1} & \vdots & w_{n_1+1} & \cdots & w_n \end{bmatrix} \begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix}$$

and we want to prove that $A_4 = 0$, $B_2 = 0$.

Controllability Reduction (cont.)

This follows from the following facts

$$
\begin{aligned}
A\mathcal{R} \subset \mathcal{R} \quad &: \quad Av_i = \alpha_1^i v_1 + \alpha_2^i v_2 + \cdots + \alpha_{n_1}^i v_{n_1}, \qquad i = 1, 2, \cdots, n \\
\mathcal{B} \subset \mathcal{R} \quad &: \quad b_j = \beta_1^j v_1 + \beta_2^j v_2 + \cdots + \beta_{n_1}^j v_{n_1}, \qquad j = 1, 2, \cdots, n
\end{aligned}
$$

established in the second lemma.

Controllability Reduction (cont.)

Therefore, if a realization $\{A, B, C, D\}$ is given with rank$[R] = n_1 < n$, We can apply

1. a coordinate transformation so that

$$\begin{aligned} A_n &= T^{-1}AT = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \qquad B_n = T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ C_n &= CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \qquad D_n = D \end{aligned}$$

2. use the fact

$$C(sI - A)^{-1}B + D = C_n(sI - A_n)^{-1}B_n + D_n$$

$$\begin{aligned} &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} (sI - A_1)^{-1} & -(sI - A_1)^{-1}A_3(sI - A_2)^{-1} \\ 0 & (sI - A_2)^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + D \\ &= C_1(sI - A_1)^{-1}B_1 + D \quad \text{(see the following remark)} \end{aligned}$$

to get the lower order realization of order n_1, which is moreover controllable.

Controllability Reduction (cont.)

Remark[2]

1. When A^{-1} and B^{-1} exist,

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{bmatrix}$$

   and

$$\begin{bmatrix} A & D \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}DB^{-1} \\ 0 & B^{-1} \end{bmatrix}.$$

2. If A^{-1} exists,

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix}$$

   where

$$\Delta = B - CA^{-1}D, \quad E = A^{-1}D, \quad F = CA^{-1}.$$

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[2] T. Kailath, *Linear Systems*, Prentice-Hall, 1980, p.656

Observability Reduction

Define

$$O := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

and let θ be the null space (or kernel) of O.

$$\theta := \{x \;:\; Ox = 0\}.$$

Obviously, θ is the subspace that is orthogonal to all the rows of O. If $\text{rank}[O] = n_2$, then θ has dimension $n - n_2$.

Observability Reduction (cont.)

Lemma

θ is A-invariant and is contained in Kernel(C). In fact, θ is the largest such subspace.

Observability Reduction (cont.)

Proof

If $v \in \theta$, then $CA^i v = 0$, $i = 0,1,\cdots,n-1$. Then $CA^j Av = 0$, $j = 0,1,\cdots,n-2$. To complete the proof of A-invariance we need to show that $CA^{n-1}Av = 0$. This follows from the Cayley-Hamilton Theorem. If $v \in \theta$, then certainly $Cv = 0$ so that

$$\theta \subset \mathrm{Kernel}(C).$$

To prove that θ is the largest such subspace, suppose that it is not and θ_1 is a larger subspace with the property

$$\theta \subset \theta_1 \subset \mathrm{Kernel}(C).$$

Then it is possible to argue and show that

$$\theta \subset \theta_1 \subset \theta.$$

Observability Reduction (cont.)

Now suppose that $\{v_1, \cdots, v_{n_2}\}$ is a basis for θ and choose $\{w_{n_2+1}, \cdots, w_n\}$ so that

$$T := \begin{bmatrix} v_1 & \cdots & v_{n_2} & w_{n_2+1} & \cdots & w_n \end{bmatrix}$$

is an invertible $n \times n$ matrix.

Observability Reduction (cont.)

Then we have the following:

Lemma

$$T^{-1}AT = \begin{bmatrix} A_1 & \vdots & A_3 \\ \cdots & & \cdots \\ 0 & \vdots & A_2 \end{bmatrix} \qquad CT = \begin{bmatrix} 0 & \vdots & C_2 \end{bmatrix} \tag{2}$$

where $A_2 \in \mathbb{R}^{n_2 \times n_2}$, $C_2 \in \mathbb{R}^{m \times n_2}$, and (C_2, A_2) observable. This pair is called the Kalman observable canonical form.

Observability Reduction (cont.)

Proof

Again Eq. (2) is equivalent to the following matrix equations.

$$A \begin{bmatrix} v_1 & \cdots & v_{n_2} & w_{n_2+1} & \cdots & w_n \end{bmatrix} =$$

$$\begin{bmatrix} v_1 & \cdots & v_{n_2} & w_{n_2+1} & \cdots & w_n \end{bmatrix} \begin{bmatrix} A_1 & A_3 \\ A_4 & A_2 \end{bmatrix}$$

$$C \begin{bmatrix} v_1 & \cdots & v_{n_2} & w_{n_2+1} & \cdots & w_n \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

and we need to show that i) $A_4 = 0$, ii) $C_1 = 0$. But this follows from

1. A-invariance of θ, $Av_i = \sum_j^{n_2} \alpha_j^i v_j$ and
2. $\theta \subset \mathrm{Kernel}(C)$ which means $Cv_i = 0$, $i = 1, 2, \cdots, n_2$.

Observability Reduction (cont.)

Therefore if a realization $\{A, B, C, D\}$ with $\text{rank}[O] = n_2 < n$ is given we can

1. apply a coordinate transformation T so that

$$
\begin{aligned}
A_n &= T^{-1}AT = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \qquad B_n = T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\
C_n &= CT = \begin{bmatrix} 0 & C_2 \end{bmatrix} \qquad D_n = D
\end{aligned}
$$

2. use the fact

$$
\begin{aligned}
C(sI - A)^{-1}B + D &= C_n(sI - A_n)^{-1}B_n + D \\
&= C_2(sI - A_2)^{-1}B_2 + D
\end{aligned}
$$

to get a realization of order $n - n_2$, which is observable.

Joint Reduction

Suppose that we have a realization (A, B, C, D) with $\text{rank}[R] = n_1$. By applying the controllability reduction we get a realization (A_1, B_1, C_1, D) of order n_1 and (A_1, B_1) is controllable. If (C_1, A_1) is observable, we are through as we have a *controllable and observable* realization. Otherwise carry out an observability reduction so that

$$\begin{aligned} T^{-1}A_1T &= \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{12} \end{bmatrix} \qquad T^{-1}B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} \\ CT &= \begin{bmatrix} 0 & C_{12} \end{bmatrix} \qquad D = D \end{aligned}$$

and we have a realization $(A_{12}, B_{12}, C_{12}, D)$ which is observable.

The question that arises is: Is (A_{12}, B_{12}) controllable?

Joint Reduction (cont.)

The answer is: If (A_1, B_1) is controllable, so is (A_{12}, B_{12}).

Remark

This shows that a two step procedure is enough to produce a controllable and observable realization (minimal realization).

Gilbert Realization

Gilbert's Realization is a particular minimal realization which can be obtained directly from a transfer function matrix $G(s)$. However, this realization is possible **only when each entry of $G(s)$ has distinct poles**.

1. Expand each entry of $G(s)$ into partial fractions.
2. Form
$$G(s) = \frac{[R_1]}{s-\alpha_1} + \frac{[R_2]}{s-\alpha_2} + \frac{[R_3]}{s-\alpha_3} + \cdots .$$
3. Total size of realization is
$$n^* = \sum_i \mathrm{Rank}\,[R_i] .$$
4. Find B_i and C_i so that
$$C_i B_i = R_i \qquad \text{where} \quad C_i \in \mathbb{R}^{n\times \mathrm{m}}, \;\; B_i \in \mathbb{R}^{m\times \mathrm{n}}; \;\; m = \mathrm{Rank}\,[R_i] .$$

Gilbert Realization (cont.)

5. Form (A, B, C) where

$$A = \begin{bmatrix} \alpha_1 I_1 & & \\ & \alpha_2 I_2 & \\ & & \ddots \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \end{bmatrix} \quad C = \begin{bmatrix} C_1 & C_2 & \cdots \end{bmatrix}.$$

Note that I_i is the identity matrix with dimension being equal to $\mathrm{Rank}\,[R_i]$.

Gilbert Realization (cont.)

Example

Find a minimal realization of the following transfer function.

$$G(s) = \begin{bmatrix} \dfrac{1}{(s-1)(s-2)} & \dfrac{1}{(s-2)(s-3)} \\ \dfrac{1}{(s-2)(s-3)} & \dfrac{1}{(s-1)(s-2)} \end{bmatrix}$$

Gilbert Realization (cont.)

Since all entries of $G(s)$ have simple poles, we can use Gilbert Realization.

$$
\begin{aligned}
G(s) &= \begin{bmatrix} \frac{-1}{s-1} + \frac{1}{s-2} & \frac{-1}{s-2} + \frac{1}{s-3} \\ \frac{-1}{s-2} + \frac{1}{s-3} & \frac{-1}{s-1} + \frac{1}{s-2} \end{bmatrix} \\
&= \frac{1}{s-1}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{s-2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{s-3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{1}{s-1}\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{C_1}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B_1} + \frac{1}{s-2}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{C_2}\underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{B_2} \\
&\quad + \frac{1}{s-3}\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{C_3}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B_3}
\end{aligned}
$$

Gilbert Realization (cont.)

Therefore,

$$A = \begin{bmatrix} 1 & 0 & \vdots & 0 & & 0 & 0 \\ 0 & 1 & \vdots & 0 & & 0 & 0 \\ & \cdots & & \cdots & & & \\ 0 & 0 & \vdots & 2 & \vdots & 0 & 0 \\ & & & \cdots & & \cdots & \cdots \\ 0 & 0 & & 0 & \vdots & -3 & 0 \\ 0 & 0 & & 0 & \vdots & 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \cdots & \cdots \\ 1 & -1 \\ \cdots & \cdots \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & \vdots & 1 & \vdots & 0 & 1 \\ 0 & -1 & \vdots & -1 & \vdots & 1 & 0 \end{bmatrix}.$$

Balanced Realizations

Recall the fact that a transfer function can be realized by infinite number of state space realizations. Depending on purpose, a designer chooses a different state space realization to implement.

One type of realization we often see is a companion form realization which is known to be highly numerically sensitive. Here we discuss another type of realization that is called a balanced realization. To proceed, we limit the scope with stable and minimal realizations.

Balanced Realizations (cont.)

Consider a stable and minimal realization of form:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t).$$

Then

$$\begin{aligned} AW_c + W_c A^T &= -BB^T \\ A^T W_o + W_o A &= -C^T C \end{aligned}$$

where the controlability Gramian W_c and the obervability Gramian W_o are positive definite.

Balanced Realizations (cont.)

Theorem

Suppose that two different state space realizations (A, B, C) *and* $(\hat{A}, \hat{B}, \hat{C})$ *are minimal and equivalent. Let* $W_c W_o$ *and* $\hat{W}_c \hat{W}_o$ *be the products of their controllability Gramian and observability Gramian, respectively. Then* $W_c W_o$ *and* $\hat{W}_c \hat{W}_o$ *are similar and positive definite.*

Balanced Realizations (cont.)

Proof

Write

$$\hat{A} = T^{-1}AT, \quad \hat{B} = T^{-1}B, \quad \hat{C} = CT.$$

Then

$$\hat{A}\hat{W}_c + \hat{W}_c\hat{A}^T = -\hat{B}\hat{B}^T$$

yields

$$\begin{aligned}
T^{-1}AT\hat{W}_c + \hat{W}_c T^T A^T T^{-T} &= -T^{-1}BB^T T^T \\
AT\hat{W}_c + T\hat{W}_c T^T A^T T^{-T} &= -T^{-1}BB^T T^{-T} \\
A\underbrace{T\hat{W}_c T^T}_{W_c} + \underbrace{T\hat{W}_c T^T}_{W_c} A^T &= -T^{-1}BB^T.
\end{aligned}$$

Balanced Realizations (cont.)

Thus, we have

$$W_c = T \hat{W}_c T^T$$

and similarly,

$$W_o = T^{-T} \hat{W}_o T^{-1}.$$

Now,

$$W_c W_o = T \hat{W}_c T^T T^{-T} \hat{W}_o T^{-1} = T \hat{W}_c \hat{W}_o T^{-1}$$

which implies that $W_c W_o$ and $\hat{W}_c \hat{W}_o$ are similar.

Balanced Realizations (cont.)

To prove they are positive definite, we need the following lemma.

Lemma

For every real symmetric matrix A, there exists an orthogonal matrix Q such that

$$A = Q^T D Q$$

where D is a diagonal matrix with the eigenvalues of A which are real.

Balanced Realizations (cont.)

Note that W_c is symmetric positive definite. Since its eigenvalues are real and positive, we write

$$W_c = Q^T D^{\frac{1}{2}} D^{\frac{1}{2}} Q =: R^T R$$

where Q is orthogonal, i.e., $Q^{-1} = Q^T$, and $R = D^{\frac{1}{2}} Q$. Consider

$$\begin{aligned} \det(sI - W_c W_o) &= \det(sI - R^T R W_o) = \det\left[R^T(sR - RW_o)\right] \\ &= \det(sI - RW_o R^T) \end{aligned}$$

which implies that the matrices $W_c W_o$ and $RW_o R^T$ have the same eigenvalues. Here, note that $RW_o R^T$ is symmetric and positive definite, therefore, so does $W_c W_o$.

Balanced Realizations (cont.)

Theorem (A Balanced Realization)

For any minimal realizaton (A, B, C), *there exists a similarity transformation such that the controllability Gramian* W_c *and observability Gramian* W_o *of its equivalent state space realization have the propoety*

$$\hat{W}_c = \hat{W}_o = \Sigma.$$

Such a equivalent realization is called a balanced realization.

Balanced Realizations (cont.)

Proof

Recall the expression RW_oR^T where

$$W_c = R^T R, \quad R = D^{\frac{1}{2}} Q.$$

Since RW_oR^T is symmetric, we can write

$$RW_oR^T = U\Sigma^2 U^T$$

where U is orthogonal.

Balanced Realizations (cont.)

Then we can write

$$U^T R W_o R^T U = \Sigma^{\frac{1}{2}} \Sigma \Sigma^{\frac{1}{2}}$$

and with (32)

$$\underbrace{\Sigma^{-\frac{1}{2}} U^T R}_{T^{-T}} W_o \underbrace{R^T U \Sigma^{-\frac{1}{2}}}_{T^{-1}} = \Sigma =: \hat{W}_o$$

Similarly, with (32)

$$\begin{aligned} \underbrace{\Sigma^{\frac{1}{2}} U^T R^{-T}}_{T} W_c \underbrace{R^{-1} U \Sigma^{\frac{1}{2}}}_{T^T} &= \Sigma^{\frac{1}{2}} U^T R^{-T} R^T R R^{-1} U \Sigma^{\frac{1}{2}} \\ &= \Sigma =: \hat{W}_c. \end{aligned}$$

Degree of Transfer Function Matrices

Definition

In a proper rational matrix $G(s)$, the characteristic polynomial of $G(s)$ is defined as the least common denominator of all minors of $G(s)$. The degree of the characteristic polynomial is called the McMillan degree.

Degree of Transfer Function Matrices (cont.)

Example

Consider

$$G_1(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+1} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix}.$$

The miniors of order 1 are all $\frac{1}{s+1}$ and the minior of order 2 is 0. The characteristic polynomial is $\delta_1(s) = s + 1$ and the McMillian degree is 1.

Degree of Transfer Function Matrices (cont.)

Consider

$$G_2(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{1}{s+1} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix}.$$

The miniors of order 1 are $\frac{1}{s+1}$ and $\frac{2}{s+1}$, and the minior of order 2 is $\frac{1}{(s+1)^2}$. So the characteristic polynomial is $\delta_2(s) = (s+1)^2$ and the McMillan degree is 2.