ECEN 605

LINEAR SYSTEMS

Lecture 14

State Feedback and Observers II
– Observer Theory

Output Feedback Stabilization

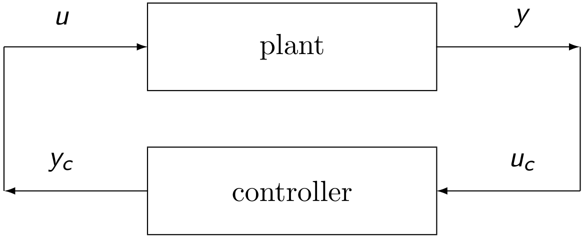

Figure 1: A Closed Loop Control System

Output Feedback Stabilization (cont.)

Consider the closed loop control system with the plant and controller described in their respective state space representations.

$$
\begin{aligned}
\text{Plant} \quad : \qquad & \dot{x} = Ax + Bu, \qquad x - n \text{ vector} \\
& y = Cx
\end{aligned}
$$

$$
\begin{aligned}
\text{Controller} \quad : \qquad & \dot{x}_c = A_c x_c + B_c u_c, \qquad x_c - n_c \text{ vector} \\
& y_c = C_c x_c + D_c u_c
\end{aligned}
$$

Output Feedback Stabilization (cont.)

Then,

$$\text{Feedback Connection} \quad : \quad \begin{aligned} u_c &= y \\ u &= y_c \end{aligned}$$

and

$$\text{Closed Loop System} \quad : \quad \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}}_{A_{cl} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}} \begin{bmatrix} x \\ x_c \end{bmatrix}.$$

Output Feedback Stabilization (cont.)

Is the closed loop system stable?

Consider the stability of

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned}$$

Definition

- Internal Stability: When $u(t) = 0$, $x(t) \longrightarrow 0$.
- External Stability: When $u(t) = 0$, $y(t) \longrightarrow 0$.

Output Feedback Stabilization (cont.)

Internal stability is stronger than and implies external stability, since if $x(t) \to 0$, then clearly $y(t) = Cx(t) \to 0$. The converse is not true in general. However, if the system is controllable and observable, then external stability implies internal stability.

Remark

- *Checking Internal Stability:* $\lambda(A) \subset \mathrm{LHP}.$
- *Checking External Stability:* *Poles of $G(s)$ in LHP.*

Output Feedback Stabilization (cont.)

Now let us get back to the closed loop system. What we really want is the internal stability of the closed loop system.

Remark

A closed loop system is internally stable iff all eigenvalues of A_{cl} lie in the open LHP.

Output Feedback Stabilization (cont.)

Basic Result

If (A, B, C) is controllable and observable, then a controller of a high enough order n_c can always be found to assign the eigenvalues of A_{cl} *arbitrarily*.

In particular, a controller of order $n-1$ *always* suffices for a plant of order n.

Plant:
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Controller:
$$\dot{x}_c = A_c x_c + B_c y$$
$$u = C_c x_c + D_c y$$

Output Feedback Stabilization (cont.)

The closed loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x \\ x_c \end{bmatrix}$$

Problem : Given (A, B, C) find (A_c, B_c, D_c, D_c) so that A_{cl} is stable, i.e., $\lambda(A_{cl}) \subset \mathbb{C}^-$.

Theorem

There exists a stabilizing controller if and only if (A, B) is stabilizable and (C, A) is detectable.

State Feedback

A quick recap of the state feedback results:

Theorem

Pole Placement Theorem: Wonham, 1967

If (A, B) is controllable, there exists F so that $\lambda(A + BF)$ is equal to any set of n prescribed eigenvalues (in conjugate pairs).

State Feedback (cont.)

Proof

a) It is true for $B = b$ (single input case). Let

$$L = \begin{bmatrix} b & Ab & A^{n-1}b \end{bmatrix}$$

and let q' be the last row of L^{-1} and

$$Q = \begin{bmatrix} q' \\ q'A \\ \vdots \\ q'A^{n-1} \end{bmatrix}.$$

Take $T = Q^{-1}$. Then

$$\hat{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad \hat{b}_n = T^{-1}b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

State Feedback (cont.)

Let

$$\hat{f} = \begin{bmatrix} \hat{f}_1 & \hat{f}_2 & \cdots & \hat{f}_n \end{bmatrix}.$$

Then we have

$$\hat{A} + \hat{b}\hat{f} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 1 \\ -a_0 + \hat{f}_1 & -a_1 + \hat{f}_2 & -a_2 + \hat{f}_3 & \cdots & -a_{n-1} + \hat{f}_n \end{bmatrix}.$$

$$\begin{aligned} |sI - \hat{A} - \hat{b}\hat{f}| &= s^n + (a_{n-1} - \hat{f}_n)s^{n-1} + \cdots + (a_0 - \hat{f}_1) \\ &= s^n + a_{n-1}^d s^{n-1} + a_{n-2}^d s^{n-2} + \cdots + a_0^d \quad \text{(arbitrary)}. \end{aligned}$$

Then $\hat{f}T^{-1} = f$ gives $A + bf$ the same characteristic polynomial.

State Feedback (cont.)

b) The multi input case can be proved by reducing to the single input case via the following lemma. The proof of this lemma is omitted.

Lemma

If (A, B) is controllable and $g \neq 0$ is any vector, then there exists F_0 such that $(A + BF_0, Bg)$ is controllable.

State Feedback (cont.)

In practice, an arbitrary F_0 will "almost always" work! Let

$$A_0 = A + BF_0, \quad b_0 = Bg.$$

Then find f_0 so that $A_0 + b_0 f_0$ has the desired eigenvalues. Then

$$F = F_0 + g f_0$$

is the state feedback so that $A + BF$ has the desired eigenvalues.

Full Order Observers

To proceed with our construction of the feedback controller we need to bring in the concept of an "observer".

Problem: Design a device which will "measure" x (n dimensional state vector) from external measurements y (m vector) and u (r vector).

Full Order Observers (cont.)

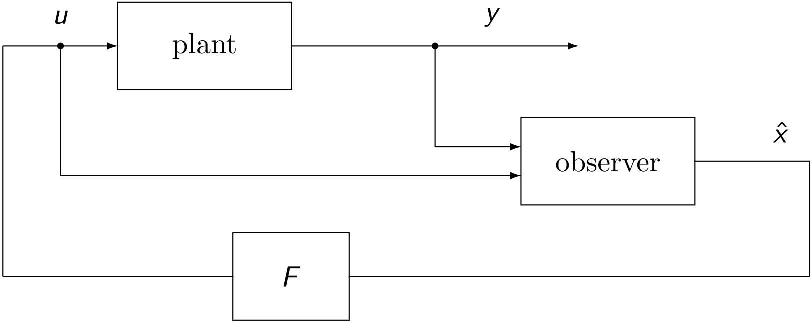

Figure 2: A Feedback with State Estimator

Full Order Observers (cont.)

System:
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Observer:
$$\dot{z} = Mz + Ly + Gu$$
$$\hat{x} = Pz + Qy + Ru$$

Requirement Design (M, L, G, P, Q, R) so that

$$\lim_{t\to\infty} (\hat{x}(t) - x(t)) = 0 \qquad \forall \;\; x(0), z(0), u(t).$$

It will turn out that this will be possible if (C, A) is detectable.

Full Order Observers (cont.)

Let $P = I_n$, $Q = R = 0$, then we have

$$\hat{x} = z.$$

Let

$$e = z - x.$$

Then

$$\begin{aligned}
\dot{e} &= \dot{z} - \dot{x} \\
&= \underbrace{Mz + Ly + Gu}_{\dot{z}} - \underbrace{(Ax + Bu)}_{\dot{x}} \\
&= Mz + LCx + Gu - Ax - Bu \\
&= Me + (M - A + LC)x + (G - B)u.
\end{aligned}$$

Full Order Observers (cont.)

Therefore, if we set

$$\begin{aligned} G &= B \\ M &= A - LC \end{aligned}$$

so that

$$\dot{e} = Me = (A - LC)e$$

and the influence of x and u on e are cancelled. For convergence of $e(t) \to 0$, we need that

$$\lambda(A - LC) \subset \mathbb{C}^-.$$

If (C, A) observable, we can find L to place $\lambda(A - LC)$ arbitrarily by the pole placement theorem.

Full Order Observers (cont.)

Remark

The eigenvalues of $(A - LC)$ *is identical to the eigenvalues of* $(A^T - C^T L^T)$*.* (C, A) *is observable iff* (A^T, C^T) *is controllable.*

The observer designed above is sometimes called an identity observer because each component of z estimates the corresponding component of x.

Minimal Order Observers

Since some of the n system states are measurable in the form of y, it should be possible to estimate only the remaining $(n - m)$ "unmeasurable" states. Without loss of generality (change coordinates if necessary), let

$$y = x_1$$

the first m components of the state vector and let x_2 be the remaining $n - m$ components which are not measurable.

Minimal Order Observers (cont.)

Then the system equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = x_1 = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or since x_2 is the state to be estimated, write

$$\dot{x}_2 = A_{22}x_2 + (B_2 u + A_{21} y) \quad (1)$$

$$A_{12}x_2 = \dot{y} - A_{11} y - B_1 u. \quad (2)$$

Minimal Order Observers (cont.)

Think of (1) as the dynamic equation for x_2 and (2) as the measurement equation for x_2.

Lemma

If (C, A) is observable then so is (A_{12}, A_{22}).

Now we apply the full order observer formulas

$$\dot{z} = (A - LC)z + Ly + Bu,$$

to x_2 and we have

$$\dot{z}_2 = (A_{22} - L_2 A_{12})z_2 + L_2(\dot{y} - A_{11}y - B_1 u) + (B_2 u + A_{21}y) \quad (3)$$

and the error will satisfy

$$\begin{aligned} e_2 &= z_2 - x_2 \\ \dot{e}_2 &= (A_{22} - L_2 A_{12})e_2. \end{aligned}$$

Minimal Order Observers (cont.)

The same as the case of full order observer, for $e(t) \to 0$, it is necessary that $\lambda(A_{22} - L_2 A_{12}) \subset \mathbb{C}^-$ by a choice of L_2. Now let us eliminate $\dot{y}$ to write the reduced order observer equation. Write

$$
\begin{aligned}
\dot{z}_2 - L_2 \dot{y} &= (A_{22} - L_2 A_{12}) z_2 + L_2(-A_{11} y - B_1 u) + (B_2 u + B_{21} y) \\
&= (A_{22} - L_2 A_{12})(z_2 - L_2 y) + (A_{22} - L_2 A_{12}) L_2 y - L_2 A_{11} y - L_2 B_2 u + A_{21} y \\
&= (A_{22} - L_2 A_{12})(z_2 - L_2 y) + \left[(A_{22} - L_2 A_{12}) L_2 - L_2 A_{11} + A_{21}\right] y + (B_2 - L_2 B_1) u.
\end{aligned}
$$

Let

$$w = z_2 - L_2 y.$$

Then eq. (3) is rewritten as follows:

$$
\begin{aligned}
\dot{w} &= (A_{22} - L_2 A_{12}) w + \left[(A_{22} - L_2 A_{12}) L_2 - L_2 A_{11} + A_{21}\right] y + (B_2 - L_2 B_1) u \\
z_2 &= w + L_2 y.
\end{aligned}
$$