# ECEN 605

## LINEAR SYSTEMS

Lecture 15

State Feedback and Observers III
– Observer Based Feedback

# Closed-lop Eigenvalues

Suppose we calculate a state feedback $u = Fx$ to place the eigenvalues of $A + BF$. Now we design an observer with eigenvalues of $M$ chosen by us. From the observer we obtain $\hat{x}$. What if we close the loop with

$$u = F\hat{x}$$

instead of $u = Fx$? What are the closed loop eigenvalues? This answer is given by the so-called *Separation Principle*.

# Closed-lop Eigenvalues (cont.)

Theorem

*The closed loop system under observed state feedback has eigenvalues*

$$\lambda(A + BF) \cup \lambda(M).$$

# Closed-lop Eigenvalues (cont.)

**(Proof of Separation Principle)** Consider the general observer

$$\dot{z} = Mz + Ly + Gu.$$

Let

$$z - Tx = e.$$

Then

$$\dot{e} = Me + (MT - TA + LC)x + (G - TB)u.$$

Therefore to make $z$ estimate $Tx$, we want $e(t) \to 0$. Thus, set

$$\begin{aligned} MT - TA + LC &= 0 \quad (1) \\ G - TB &= 0 \quad (2) \end{aligned}$$

and

$$\lambda(M) \subset \mathbb{C}^-.$$

# Closed-lop Eigenvalues (cont.)

To estimate $Fx$ we use the observer output

$$\begin{aligned} t := F\hat{x} &= S_1 z + S_2 y \\ &= (S_1 T + S_2 C)x + S_1 e \end{aligned}$$

and set

$$S_1 T + S_2 C = F. \tag{3}$$

The loop is closed by setting $u = t$ (instead of $u = Fx$). Now look at the closed loop equations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{z} &= Mz + Ly + Gu \\ u &= S_1 z + S_2 y. \end{aligned}$$

We have

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} A + BS_2 C & BS_1 \\ LC + GS_2 C & M + GS_1 \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x \\ z \end{bmatrix}. \tag{4}$$

# Closed-lop Eigenvalues (cont.)

Now make the coordinate transformation

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$
$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I & 0 \\ -T & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.$$

Then using (1) - (4), we have

$$\begin{bmatrix} I & 0 \\ -T & I \end{bmatrix} A_{cl} \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} = \begin{bmatrix} A + BF & BS_1 \\ 0 & M \end{bmatrix}$$

Therefore,

$$\lambda(A_{cl}) = \lambda(A + BF) \cup \lambda(M).$$

# Closed-lop Eigenvalues (cont.)

The transfer function of the observer as controller is derived as follows:

$$\begin{aligned}\dot{z} &= Mz + Ly + Gu \\ t &= S_1 z + S_2 y \\ \dot{z} &= (M + GS_1)z + (L + GS_2)y\end{aligned}$$

Therefore,

$$t(s) = \underbrace{\left[S_1(sI - M - GS_1)^{-1}(L + GS_2) + S_2\right]}_{C(s)} y(s)$$

□

# Closed-lop Eigenvalues (cont.)

## Example

Consider the following single-input two-output system:

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x
\end{aligned}
$$

Design a full order observer with eigenvalues at $-1$, $-2$, $-3$ to estimate the state $x$.

# Closed-lop Eigenvalues (cont.)

**Solution**

$$A - LC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \\ l_{31} & l_{32} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -l_{11} & 1 & -l_{12} \\ -l_{21} & 0 & -l_{22} \\ 1 - l_{31} & 0 & -l_{32} \end{bmatrix}$$

One easy choice is:

$$l_{31} = 1, \quad l_{32} = 3, \quad s^2 + l_{11}s + l_{21} = s^2 + 3s + 2$$

Thus, we have

$$l_{11} = 3, \quad l_{21} = 2, \quad l_{12}, l_{22} \text{ arbitrary}, \text{ say}, \ 0.$$

# Closed-lop Eigenvalues (cont.)

Then

$$L = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 3 \end{bmatrix}$$

and

$$A - LC = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Therefore, the observer is

$$\dot{z} = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} z + \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 3 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$z = \hat{x}$ is the estimate of $x$.

# Closed-lop Eigenvalues (cont.)

## Example

For the system given in the previous example, design a minimal order observer with eigenvalue at $-3$.

# Closed-lop Eigenvalues (cont.)

**Solution**

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} := \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Now

$$\begin{aligned} \dot{\bar{x}}_1 &= \dot{x}_1 = x_2 + u = \bar{x}_3 + u \\ \dot{\bar{x}}_2 &= \dot{x}_3 = x_1 + u = \bar{x}_1 + u \\ \dot{\bar{x}}_3 &= \dot{x}_2 = 0 \\ &\Downarrow \end{aligned}$$

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \vdots & 1 \\ 1 & 0 & \vdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$$

# Closed-lop Eigenvalues (cont.)

Thus,

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad A_{22} = 0 \qquad B_2 = 0$$

The observer is:

$$\dot{w}_2 = (A_{22} - L_2 A_{12}) w_2 + [(A_{22} - L_2 A_{12}) L_2 - L_2 A_{11} + A_{21}] y + (B_2 - L_2 B_1) u$$
$$\hat{\hat{x}}_3 = w_2 - L_2 y.$$

Since the eigenvalue of $A_{22} - L_2 A_{12}$ should be $-3$, we have

$$0 - \begin{bmatrix} l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} := -3$$

Here, we choose $l_{21} = 3$ and $l_{22} = 0$. Therefore,

$$\begin{aligned} \dot{w}_2 &= -3w_2 - 9y_1 - 3u \\ \hat{\hat{x}}_3 &= w_2 + 3y_1 = \hat{x}_2. \end{aligned}$$

# Closed-lop Eigenvalues (cont.)

## Example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

i) Design a minimal order state estimator with its pole at $-1$.

$$\dot{x} = Ax + bu \qquad y = cx$$

We need to pick $T$ so that

$$cV = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We choose

$$V = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

# Closed-lop Eigenvalues (cont.)

So, we have new coordinate system with $V$ as follows:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \qquad y = \bar{c}\bar{x}$$

where

$$\bar{A} = V^{-1}AV = \begin{bmatrix} 0 & \vdots & -1 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \qquad \bar{b} = V^{-1}B = \begin{bmatrix} 1 \\ \cdots \\ -1 \end{bmatrix}$$

$$\bar{c} = cV = \begin{bmatrix} 1 & \vdots & 0 \end{bmatrix}.$$

The minimal order state estimator equation is

$$\dot{w} = (\bar{A}_{22} - L_2\bar{A}_{12})w + \left[(\bar{A}_{22} - L_2\bar{A}_{12})L_2 + L_2\bar{A}_{11} + \bar{A}_{21}\right] y + (\bar{b}_2 - L_2\bar{b}_1)u$$

# Closed-lop Eigenvalues (cont.)

Let us select $L_2$ such that

$$\lambda(\bar{A}_{22} - L_2\bar{A}_{12}) = -1.$$

Then we have $L_2 = -1$. Therefore, the minimal order state estimator becomes

$$\begin{aligned} \dot{w} &= -w + y \\ \hat{\bar{x}}_2 &= w + L_2 y. \end{aligned}$$

# Closed-lop Eigenvalues (cont.)

ii) Combine the state feedback and the observer into an output feedback controllor, and give the state equations as well as the transfer function of this controller. Choose a state feedback to assign poles of the original system at all $-1$.

Now we need to consider the coordinate transformation.

$$\hat{x} = V\hat{\bar{x}}, \quad y = \bar{c}\bar{x}, \quad w = \hat{\bar{x}}_2 - L_2 y$$

$$\left[\begin{array}{c} y \\ w \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ -L_2 & 1 \end{array}\right] \underbrace{\left[\begin{array}{c} \bar{x}_1 \\ \hat{\bar{x}}_2 \end{array}\right]}_{\hat{\bar{x}}}.$$

Thus,

$$\begin{aligned} \hat{x} &= V\hat{\bar{x}} = V \left[\begin{array}{cc} 1 & 0 \\ L_2 & 1 \end{array}\right] \left[\begin{array}{c} y \\ w \end{array}\right] \\ &= \left[\begin{array}{cc} 1 & 0 \\ 1 & -1 \end{array}\right] \left[\begin{array}{c} y \\ w \end{array}\right] \end{aligned}$$

# Closed-lop Eigenvalues (cont.)

From the control law, $f = [f_1 \quad f_2] = [-1 \quad -2]$ which assigns the original poles at $-1$, we have

$$\begin{aligned} u = F\hat{x} &= \begin{bmatrix} -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} \\ &= -3y + 2w. \end{aligned}$$

Therefore, the controller state equations are:

$$\begin{aligned} \dot{w} &= -w + y \\ u &= F\hat{x} = 2w - 3y \end{aligned}$$

# Closed-lop Eigenvalues (cont.)

and the controller transfer function is:

$$\begin{aligned} G(s) &= c(sI - A)^{-1}b + d \\ &= 2(s+1)^{-1}1 + (-3) \\ &= \frac{2}{s+1} - 3 \\ &= \frac{-3s-1}{s+1}. \end{aligned}$$

# Closed-lop Eigenvalues (cont.)

## Example

Consider a system

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let a full order observer be

$$\begin{aligned} \dot{z} &= Mz + Ly + Gu \\ \hat{x} &= Pz + Qy. \end{aligned}$$

We choose the observer poles

$$\Lambda_o = \{-0.1, -0.2, -0.3\},$$

# Closed-lop Eigenvalues (cont.)

then from $M = A - LC$, we have

$$L = \begin{bmatrix} 1.1998 & -0.0008 \\ -0.0148 & 0.0301 \\ -0.0644 & 0.4002 \end{bmatrix}$$

and

$$M = A - LC = \begin{bmatrix} -0.1998 & 0 & 0.0008 \\ 0.0148 & 0 & -0.0301 \\ 0.0644 & 1 & -0.4002 \end{bmatrix}.$$

Thus, the observer equations are (seting $G = B$, $P = I$, $Q = R = 0$)

$$\begin{aligned} \dot{z} &= \begin{bmatrix} -0.1998 & 0 & 0.0008 \\ 0.0148 & 0 & -0.0301 \\ 0.0644 & 1 & -0.4002 \end{bmatrix} z + \begin{bmatrix} 1.1998 & -0.0008 \\ -0.0148 & 0.0301 \\ -0.0644 & 0.4002 \end{bmatrix} y + \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} u \\ \hat{x} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z. \end{aligned} \quad (5)$$

# Closed-lop Eigenvalues (cont.)

The first figure in Figure 1 shows the plots of the states $x(t)$ and the estimated states $\hat{x}(t)$. This shows all three states are estimated correctly. To close look at the convergence, we plot the error state

$$e(t) = \hat{x}(t) - x(t)$$

in the second figure of Figure 1 which shows the convergence. The initial conditions of these plot were:

$$x(0) = [0 \quad -1 \quad -2] \quad \text{and} \quad z(0) = [-5 \;\; 0 \;\; 5].$$

# Closed-lop Eigenvalues (cont.)

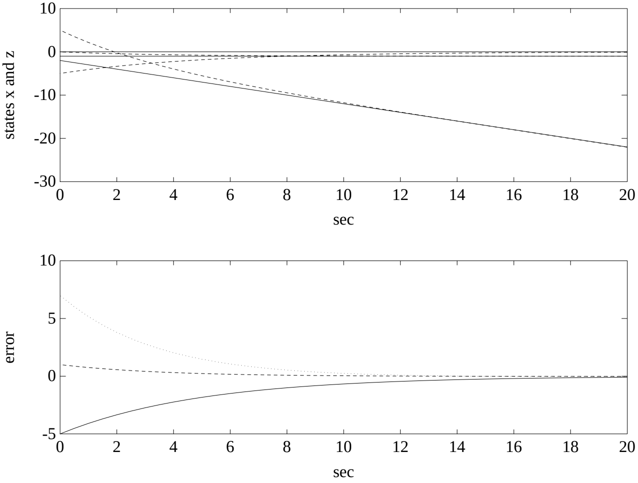


Figure 1: $x(t)$, $\hat{x}(t)$, and $e(t)$ $(x(0) = [0 \;\; -1 \;\; -2])$

# Closed-lop Eigenvalues (cont.)

Now let us choose a different initial conditions: we choose

$$x(0) = [1 \;\; 2 \;\; 3]$$

while maintaining $z(0)$ the same. The state $x(t)$ and $\hat{x}(t)$ are shown in Figure 2. The error $e(t)$ is also shown.

# Closed-lop Eigenvalues (cont.)

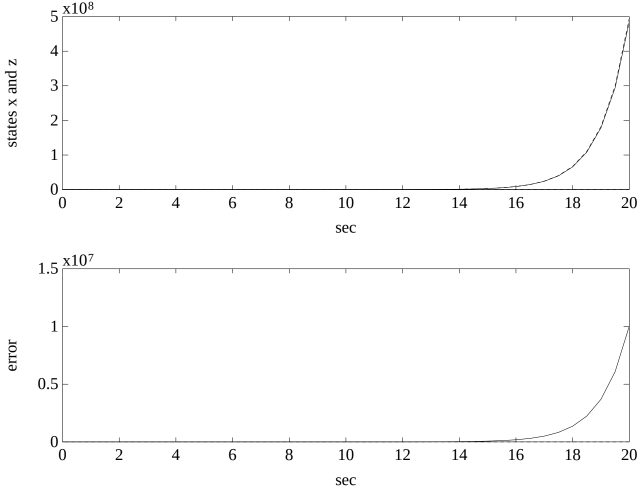


Figure 2: $x(t)$, $\hat{x}(t)$, and $e(t)$ (Problem 3: $x(0) = [1\ 2\ 3]$)

# Closed-lop Eigenvalues (cont.)

This shows that the estimated states do not converge to the true states. Obviously the full order observer we designed does not function properly. This interesting fact can be explained in terms of the robustness of the observer.