ECEN 605

LINEAR SYSTEMS

Lecture 17

State Feedback and Observers VI
– Pole Placement Compensators

Pole Placement Compensators

Let the plant equation be

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \qquad \text{measured output} \end{aligned}$$

Let the compensator equation be

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u_c &= C_c x_c + D_c y \end{aligned}$$

Therefore, the plant and compensator transfer functions are

$$\begin{aligned} G_p(s) &= C(sI - A)^{-1} B \\ G_c(s) &= C_c(sI - A_c)^{-1} B_c + D_c. \end{aligned}$$

Pole Placement Compensators (cont.)

We would like to design $(A_c,\ B_c,\ C_c,\ D_c)$.

Design Problem: Choose $(A_c,\ B_c,\ C_c,\ D_c)$ such that the closed loop eigenvalues assume the desired values. (General Pole Placement Compensators)

What are the closed loop eigenvalues?

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v$$

where $v + u_c = u$. If the compensator order is n_c, $n + n_c$ eigenvalues will be placed in the prescribed locations.

How do we find $(A_c,\ B_c,\ C_c,\ D_c)$?

Single Input System

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx \end{aligned}$$

1. Let p^* be the smallest integer j such that

$$\text{Rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^j \end{bmatrix} = n(\text{full rank})$$

Single Input System (cont.)

p^* exists and is unique if (C, A) is observable, which we assume.
Add p $(p \geq p^*)$ integrators to the system as follows:

$$
\begin{aligned}
u &= u_1 \\
\dot{u}_1 &= u_2 \\
\dot{u}_2 &= u_3 \\
&\vdots \\
\dot{u}_p &= v
\end{aligned}
$$

Introduce the extended state vector

$$
\bar{x} := \begin{bmatrix} x \\ u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}
$$

Single Input System (cont.)

and consider the augmented system

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}v$$

where

$$\bar{A} = \begin{bmatrix} A & b & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & & & \vdots \\ \vdots & & 0 & 1 & & \vdots \\ \vdots & & & & \ddots & 0 \\ \vdots & & & & & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

We note that $(\bar{A}, \bar{b})$ is controllable if (A, b) is so.

Single Input System (cont.)

2. Find the state feedback control

$$v = \bar{f}\bar{x}$$

so that $\bar{A} + \bar{b}\bar{f}$ has a prescribed set Λ of $n + p$ eigenvalues in the LHP. Write

$$\underbrace{\bar{f}}_{1\times n+p} = \left[\begin{array}{ccccc} \underbrace{f_0}_{1\times n} & f_1 & f_2 & \cdots & f_p \end{array}\right]$$

Single Input System (cont.)

3. Introduce the feedback compensator with transfer function (input y; output u).

$$C(s) = \frac{\beta_0 + \beta_1 s + \cdots + \beta_p s^p}{s^p - \alpha_{p-1}s^{p-1} - \cdots - \alpha_1 s - \alpha_0}$$

where $C(s)$ is $1 \times m$, β_i are $1 \times m$, and α_i are scalar. A state space realization of this compensator is

$$A_c = \begin{bmatrix} 0 & & & \alpha_0 \\ 1 & & & \alpha_1 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & 1 & \alpha_{p-1} \end{bmatrix} \qquad B_c = \begin{bmatrix} \beta_0 + \alpha_0\beta_p \\ \beta_1 + \alpha_1\beta_p \\ \vdots \\ \vdots \\ \beta_{p-1} + \alpha_{p-1}\beta_p \end{bmatrix}$$

$$c_c = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \qquad D_c = \beta_p.$$

Single Input System (cont.)

4. Calculate the β_i from the equation

$$\begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_p \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^p \end{bmatrix} = f_0.$$

This equation has a solution because of the observability assumption.

Single Input System (cont.)

5. Calculate the α_i from the equations

$$
\begin{aligned}
\beta_p CA^{p-1} b + \beta_{p-1} CA^{p-2} b + \cdots + \beta_1 Cb + \alpha_0 &= f_1 \\
\beta_p CA^{p-2} b + \beta_{p-1} CA^{p-3} b + \cdots + \beta_2 Cb + \alpha_1 &= f_2 \\
&\vdots \\
\beta_p Cb + \alpha_{p-1} &= f_p
\end{aligned}
$$

This determines all the compensator parameters. It remains to be proved that the closed loop poles (or eigenvalues) are exactly equal to the set Λ of $n + p$ eigenvalues that were assigned in Step 2.

Single Input System (cont.)

Theorem

The compensator designed above assigns the closed loop eigenvalues to the set Λ.

Single Input System (cont.)

Proof

We will show that the closed loop system matrix A_{cl} is similar to $\bar{A} + \bar{b}\bar{f}$. Let us write out

$$A_{cl} = \begin{bmatrix} A + bD_c C & bc_c \\ B_c C & A_c \end{bmatrix}$$

$$= \left[\begin{array}{c|cccccc} A + b\beta_p C & 0 & 0 & \cdots & 0 & b \\ (\beta_0 + \alpha_0\beta_p)C & 0 & 0 & \cdots & 0 & \alpha_0 \\ (\beta_1 + \alpha_1\beta_p)C & 1 & & & & \alpha_1 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ (\beta_{p-1} + \alpha_{p-1}\beta_p)C & 0 & \cdots & 0 & 1 & \alpha_{p-1} \end{array}\right]$$

□

Single Output System

1. Let q^* be the least integer j such that

$$\mathrm{Rank}\begin{bmatrix} B & AB & A^2B & \cdots & A^jB \end{bmatrix} = n$$

q^* exists and is unique when (A, B) is controllable. The compensator order can be any $q \geq q^*$.

Single Output System (cont.)

2. Form the pair $(\bar{c}, \bar{A})$, $\bar{A} \in \mathbb{R}^{(n+q)\times(n+q)}$, $\bar{c} \in \mathbb{R}^{1\times(n+q)}$ with

$$
\begin{aligned}
\bar{A} &= \begin{bmatrix} A & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ c & 0 & & & & & 0 \\ 0 & 1 & & & & & \vdots \\ 0 & 0 & 1 & & & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix} \\
\bar{c} &= \begin{bmatrix} \underline{0} & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$(\bar{c}, \bar{A})$ is observable when (c, A) is.

Single Output System (cont.)

3. Using pole placement, find

$$\bar{L} = \begin{bmatrix} L_0 \\ l_1 \\ \vdots \\ l_q \end{bmatrix}$$

so that $\bar{A} + \bar{L}\bar{c}$ has the desired set Λ of $n + q$ eigenvalues in the left half of the complex plane.

Single Output System (cont.)

4. Consider the feedback compensator with transfer function

$$\frac{\gamma_0 + \gamma_1 s + \cdots + \gamma_q s^q}{s^q - \alpha_{q-1} s^{q-1} - \cdots - \alpha_1 s - \alpha_0} \qquad \gamma_i \in \mathbb{R}^v, \quad \alpha_i \text{ is scalar}$$

A state space realization of this is

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & 1 \\ \alpha_0 & \alpha_1 & \cdots & \cdots & \alpha_{q-1} \end{bmatrix} \qquad b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} \gamma_0 + \alpha_0 \gamma_q & \gamma_1 + \alpha_1 \gamma_q & \cdots & \cdots & \gamma_{q-1} + \alpha_{q-1} \gamma_q \end{bmatrix} \qquad D_c = \gamma_q$$

Single Output System (cont.)

5. Calculate γ_i, $i = 0, 1, \cdots, q$ from

$$\begin{bmatrix} B & AB & \cdots & A^q B \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_q \end{bmatrix} = L_0$$

This equation has a solution because the matrix on the left has full column rank.

Single Output System (cont.)

6. Calculate the α_i, $i = 0, 1, \cdots, q-1$ from

$$
\begin{aligned}
cB\gamma_1 + cAB\gamma_2 + \cdots + cA^{q-1}B\gamma_q + \alpha_0 &= l_1 \\
cB\gamma_2 + \cdots + cA^{q-2}B\gamma_q + \alpha_1 &= l_2 \\
&\vdots \\
cB\gamma_q + \alpha_{q-1} &= l_q
\end{aligned}
$$

This determines all the compensator parameters.

Single Output System (cont.)

Theorem

The compensator designed by the above procedure assigns the closed loop eigenvalues to Λ*.*

Single Output System (cont.)

Proof

By construction we have that the eigenvalues of $\bar{A} + \bar{L}\bar{c}$ equal to Λ. We will show that $\bar{A} + \bar{L}\bar{c}$ is similar to the closed loop matrix A_{cl}. For this write out A_{cl}:

$$A_{cl} = \begin{bmatrix} A + B\gamma_q c & B(\gamma_0 + \alpha_0\gamma_q) & B(\gamma_1 + \alpha_1\gamma_q) & \cdots & \cdots & B(\gamma_{q-1} + \alpha_{q-1}\gamma_q) \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & \vdots \\ \vdots & & & & \ddots & 0 \\ \vdots & & & & & 1 \\ c & \alpha_0 & \alpha_1 & \cdots & \cdots & \alpha_{q-1} \end{bmatrix}$$

$$\bar{A} + \bar{L}\bar{c} = \begin{bmatrix} A & 0 & 0 & \cdots & 0 & L_0 \\ c & 0 & 0 & & 0 & l_1 \\ 0 & 1 & & & \vdots & l_2 \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & l_q \end{bmatrix}$$

Single Output System (cont.)

Now it can be verified that

$$QA_{cl} = (\bar{A} + \bar{L}\bar{c})Q$$

where

$$\begin{aligned} Q &= \begin{bmatrix} I_n & P_1 & P_2 & \cdots & \cdots & \cdots & P_q \\ 0 & -l_2 & -l_3 & \cdots & \cdots & -l_q & 1 \\ 0 & -l_3 & & & -l_q & 0 & 0 \\ \vdots & & & & & & \vdots \\ \vdots & -l_q & & & & & \vdots \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \\ P_i &= -B\gamma_i - AB\gamma_{i+1} - \cdots - A^{q-i}B\gamma_q, \qquad i = 1, 2, \cdots, q. \end{aligned}$$

□

Single Output System (cont.)

Example

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

Single Output System (cont.)

Since $n = 4$, we need to add $n - 1 = 3$ integrators.

$$u_1 = u, \quad \dot{u}_1 = u_2, \quad \dot{u}_2 = u_3, \quad \dot{u}_3 = v$$

$$\begin{bmatrix} \dot{x} \\ \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} A & b & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{b}} v$$

$$v = f_0 x + f_1 u_1 + f_2 u_2 + f_3 u_3 = \underbrace{\begin{bmatrix} f_0 & f_1 & f_2 & f_3 \end{bmatrix}}_{f} \begin{bmatrix} x \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

and

$$f_0 = \begin{bmatrix} f_{01} & f_{02} & f_{03} & f_{04} \end{bmatrix}.$$

Single Output System (cont.)

Now suppose we assign the closed eigenvalues at -1, -2, -3, -4, -5, -6, and -7. Then

$$L = \left[\begin{array}{ccccc} \bar{b} & \bar{A}\bar{b} & \bar{A}^2\bar{b} & \cdots & \bar{A}^6\bar{b} \end{array} \right]$$

$$L^{-1} = \left[\begin{array}{c} \\ \\ \\ \text{the last row} \end{array} \right] \quad \Leftarrow \quad q$$

$$T^{-1} = \left[\begin{array}{c} q \\ q\bar{A} \\ \vdots \\ q\bar{A}^6 \end{array} \right]$$

$$\hat{\bar{A}} = T^{-1}\bar{A}T \qquad \hat{\bar{b}} = T^{-1}\bar{b}$$

Single Output System (cont.)

$$\hat{\hat{A}} + \hat{\hat{b}}\hat{f} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hat{f}_{01} & \hat{f}_{02} & \hat{f}_{03} & 1 + \hat{f}_{04} & -2 - \hat{f}_1 & \hat{f}_2 & 2 + \hat{f}_3 \end{bmatrix}.$$

Once we find $\hat{f}$, we have

$$f = \hat{f}T^{-1} = \begin{bmatrix} f_0 & f_1 & f_2 & f_3 \end{bmatrix}.$$

Single Output System (cont.)

Now let the controller transfer function be

$$C(s) = \frac{\beta_0 + \beta_1 s + \beta_2 s^2 + \beta_3 s^3}{s^3 - \alpha_2 s^2 - \alpha_1 s - \alpha_0}$$

then solve the following matrix equation to get the numerator coefficits.

$$\begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \end{bmatrix} = \begin{bmatrix} f_{01} & f_{02} & f_{03} & f_{04} \end{bmatrix} = f_0.$$

The denominator coefficients are obtained from the following.

$$\begin{aligned} \alpha_0 + \beta_1 cb + \beta_2 cAb + \beta_3 cA^2 b &= f_1 \\ \alpha_1 + \beta_2 cb + \beta_2 cAb &= f_2 \\ \alpha_2 + \beta_3 cb &= f_3. \end{aligned}$$

Single Output System (cont.)

Finally, we have

$$C(s) = \frac{-15768 - 7466s - 12398s^2 - 12154s^3}{s^3 + 30s^2 + 382s + 2722}.$$

Single Output System (cont.)
Example

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Find a pole placement compensator so that all closed loop poles are assigned at -2.

Since

$$\mathrm{Rank} \begin{bmatrix} B & AB \end{bmatrix} = 3,$$

the order of compensator is 1. Thus, we form the argument system

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \bar{c} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

Single Output System (cont.)

Since (c, A) is observable, $(\bar{c}, \bar{A})$ is onservable. Now let us find L so that $\lambda(\bar{A} + L\bar{c}) = \Lambda$.

$$\begin{aligned}
\bar{A} + L\bar{c} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} l_{01} \\ l_{02} \\ l_{03} \\ l_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & l_{01} \\ 1 & 0 & 1 & l_{02} \\ 0 & 1 & 0 & l_{03} \\ 1 & 0 & 0 & l_1 \end{bmatrix}
\end{aligned}$$

We found

$$L = \begin{bmatrix} \left.\begin{matrix} -26 \\ -48 \\ -42 \end{matrix}\right\} L_0 \\ \left.-8\right\} L_1 \end{bmatrix}.$$

Single Output System (cont.)

Now let the compensator transfer function be

$$C(s) = \frac{\gamma_0 + \gamma_1 s}{s - \alpha_0}.$$

We solve the following matrix equation

$$\begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{11} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} -26 \\ -48 \\ -42 \end{bmatrix} = L_0.$$

Thus we have

$$\gamma_0 = \begin{bmatrix} -26 \\ -42 \end{bmatrix} \qquad \gamma_1 = \begin{bmatrix} -1 \\ -47 \end{bmatrix}.$$

Single Output System (cont.)

Now we compute α_0

$$cB\gamma_1 + \alpha_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -47 \end{bmatrix} + \alpha_0 = L_1 = -8.$$

Thus, we have

$$\alpha_0 = -7.$$

A state space representation of the compensator is

$$A_c = -7, \quad b_c = 1, \quad C_c = \gamma_0 + \alpha_0 \gamma_1 = \begin{bmatrix} -19 \\ 287 \end{bmatrix}, \quad D_c = \begin{bmatrix} -1 \\ -47 \end{bmatrix}.$$

Single Output System (cont.)

The composite system is

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & -19 \\ 1 & 0 & 1 & 0 \\ -47 & 1 & 0 & 287 \\ 1 & 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix}.$$