ECEN 605

LINEAR SYSTEMS

Lecture 18

State Feedback and Observers VII
– Minimal or Maximal Realization

Minimal or Maximal Realization

The dynamic models that arise in control engineering depend on various system parameters which are imprecisely known at best and are subject at least to small perturbations.

Nevertheless, controller design proceeds by setting these parameters to nominal values, designing a controller for the nominal system, and validating its performance over the expected uncertainty set. In the context of linear multivariable control theory, this often amounts to starting with a parametrized plant transfer matrix $P(s, \mathbf{p})$, setting $\mathbf{p} = \mathbf{p}_0$ the nominal value, constructing a minimal state space realization of $P_0(s) = P(s, \mathbf{p}_0)$ and designing a feedback controller that stabilizes this minimal realization.

Minimal or Maximal Realization (cont.)

The purpose of this lecture is to point out a potential pitfall in this procedure. Specifically, we show that there are situations in which stabilization of any minimal order state space realization of $P(s, \mathbf{p}_0)$ will lead to a closed-loop system which becomes unstable for arbitrarily small perturbations of $\mathbf{p}_0$. This occurs because the McMillan degree of $P(s, \mathbf{p})$ is, in general, discontinuous with respect to $\mathbf{p}$ at $\mathbf{p}_0$.

We also show that such a catastrophic failure can be avoided if the correct class of parametrizations to which $P_0(s)$ belongs is known. Let ν^+_{max} denote maximal McMillan degree of the antistable component of $P(s, \mathbf{p})$ under the given parametrization. It can be easily found by arbitrarily perturbing $\mathbf{p}_0$ by a small amount within its perturbation class, since the McMillan degree drops only on an algebraic variety.

Minimal or Maximal Realization (cont.)

By stabilizing a perturbed nominal plant whose minimal order realization has ν^+_{max} unstable poles, the structural instability discussed above can be overcome as the controller also stabilizes a "ball" of plants centered at the new nominal. Such a stability ball around the perturbed nominal plant however cannot include the original system if $\nu^+ < \nu^+_{max}$.

Motivation

By way of motivation, consider the transfer function

$$P_0(s) = \left[\begin{array}{c:c} \dfrac{2}{(s+1)(s-1)} & \dfrac{1}{s-1} \\ \hdashline \dfrac{1}{s-1} & \dfrac{1}{s-1} \end{array} \right]$$

which represents an unstable plant to be stabilized by feedback. The order of a minimal realization of $P_0(s)$ is 2, and it can be stabilized by a compensator C_0, say of order q.

Motivation (cont.)

Now suppose that $P_0(s)$ perturbs to

$$P_1(s) = P(s,\delta) = \left[\begin{array}{c:c} \dfrac{2}{(s+1)(s-1)} & \dfrac{1+\delta}{s-1} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \dfrac{1}{s-1} & \dfrac{1}{s-1} \end{array}\right]$$

where δ is a real parameter perturbation. It is easily seen that the closed-loop system with compensator C_0 and plant $P_1(s)$ is unstable with a closed-loop pole near $s = 1$. Moreover, this occurs for every nominally stabilizing controller C_0 of $P_0(s)$, and for infinitesimally small perturbations δ.

Motivation (cont.)

To avoid the undesirable situation discussed above, it is necessary to know the class of uncertain systems to which $P_0(s)$ belongs, and to stabilize a perturbed version of $P_0(s)$ which has the generic maximal order of unstable poles in the perturbation class.

In the example above, the nominal plant should have been chosen after perturbation ($P_1(s)$ with $\delta \neq 0$) and realized minimally to be of order 3, and a stabilizing controller C_1 designed for it. The controller C_1 remains stabilizing under small perturbations. It is important to note however that such a stability "ball" around the perturbation cannot include the nominal system $P_0(s)$!

Nominal and Parametrized Models

Consider a plant parametrized by a family of rational proper transfer function matrices, $P(s, \mathbf{p})$ where $\mathbf{p}$ is an ℓ dimensional real parameter vector which ranges over an uncertainty set $\Omega \subset \mathbb{R}^\ell$.

We assume that the coefficients of the transfer functions in $P(s, \mathbf{p})$ are continuous functions of $\mathbf{p}$ and that $P(s, \mathbf{p})$ has a state space realization $[A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p}), D(\mathbf{p})]$ with matrix entries being continuous functions of $\mathbf{p}$. Let $\mathbf{p} = \mathbf{p}_0$ be the nominal parameter and denote

$$P\left(s, \mathbf{p}_0\right) = P_0(s).$$

Nominal and Parametrized Models (cont.)

Now let $\nu[P(s, \mathbf{p})]$ denote the McMillan degree of $P(s, \mathbf{p})$.

Decompose $P(s, \mathbf{p})$ into its stable and antistable (all poles unstable) components

$$P(s, \mathbf{p}) = P^-(s, \mathbf{p}) + P^+(s, \mathbf{p})$$

and let

$$\begin{aligned} \nu\left[P^+(s, \mathbf{p})\right] &=: \nu^+[P(s, \mathbf{p})] \\ \nu\left[P^-(s, \mathbf{p})\right] &=: \nu^-[P(s, \mathbf{p})]. \end{aligned}$$

When the context is clear, we write $\nu(\mathbf{p})$, $\nu^+(\mathbf{p})$ etc. instead of $\nu[P(s, \mathbf{p}]$, $\nu^+[P(s, \mathbf{p})]$.

Nominal and Parametrized Models (cont.)

In general the functions $\nu(\mathbf{p})$, $\nu^+(\mathbf{p})$, $\nu^-(\mathbf{p})$ are discontinuous functions of $\mathbf{p}$. Moreover the generic, maximal McMillan degree depends on the specific structure of the parametrization as we show next.

Nominal and Parametrized Models (cont.)

Example

Continuing with our previous plant

$$P_0(s) = \left[\begin{array}{ccc} \dfrac{2}{(s+1)(s-1)} & \vdots & \dfrac{1}{s-1} \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \dfrac{1}{s-1} & \vdots & \dfrac{1}{s-1} \end{array} \right] \tag{1}$$

consider, for example, the four parametrized families to which $P_0(s)$ might belong:

Nominal and Parametrized Models (cont.)

$$P_1(s,\mathbf{a}) = \begin{bmatrix} \dfrac{2+a_1}{(s-1+a_2)(s+1)} & \dfrac{1+a_1}{s-1+a_2} \\ \dfrac{1+a_1}{s-1+a_2} & \dfrac{1+a_1}{s-1+a_2} \end{bmatrix}, \qquad \begin{aligned} \mathbf{a} &= [a_1 \quad a_2] \\ \mathbf{a}_0 &= [0 \quad 0] \end{aligned} \tag{2}$$

$$P_2(s,\mathbf{b}) = \begin{bmatrix} \dfrac{2+b_1}{(s+1)(s-1+b_5)} & \dfrac{1+b_2}{s-1+b_5} \\ \dfrac{1+b_3}{s-1+b_6} & \dfrac{1+b_4}{s-1+b_6} \end{bmatrix}, \qquad \begin{aligned} \mathbf{b} &= [b_1 \quad b_2 \quad \cdots \quad b_6] \\ \mathbf{b}_0 &= [0 \quad 0 \quad \cdots \quad 0] \end{aligned} \tag{3}$$

$$P_3(s,\mathbf{c}) = \begin{bmatrix} \dfrac{2+c_1}{(s+1)(s-1+c_5)} & \dfrac{1+c_2}{s-1+c_5} \\ \dfrac{1+c_3}{s-1+c_5} & \dfrac{1+c_4}{s-1+c_6} \end{bmatrix}, \qquad \begin{aligned} \mathbf{c} &= [c_1 \quad c_2 \quad \cdots \quad c_6] \\ \mathbf{c}_0 &= [0 \quad 0 \quad \cdots \quad 0] \end{aligned} \tag{4}$$

$$P_4(s,\mathbf{d}) = \begin{bmatrix} \dfrac{2+d_1}{(s+1)(s-1+d_5)} & \dfrac{1+d_2}{s-1+d_6} \\ \dfrac{1+d_3}{s-1+d_7} & \dfrac{1+d_4}{s-1+d_8} \end{bmatrix}, \qquad \begin{aligned} \mathbf{d} &= [d_1 \quad d_2 \quad \cdots \quad d_8] \\ \mathbf{d}_0 &= [0 \quad 0 \quad \cdots \quad 0] \end{aligned} \tag{5}$$

Nominal and Parametrized Models (cont.)

Write

$$\nu\left[P_i^+(s,\mathbf{a})\right] = \nu_i^+(\mathbf{a}), \quad i = 1, \cdots, 4$$

and note that the nominal transfer functions are identical

$$P_0(s) = P_1(s,\mathbf{a}_0) = P_2(s,\mathbf{b}_0) = P_3(s,\mathbf{c}_0) = P_4(s,\mathbf{d}_0),$$

with McMillan degrees all equal to 2 and antistable McMillan degrees equal to 1:

$$1 = \nu^+\left[P_0(s)\right] = \nu_1^+(\mathbf{a}_0) = \nu_2^+(\mathbf{b}_0) = \nu_3^+(\mathbf{c}_0) = \nu_4^+(\mathbf{d}_0).$$

Nominal and Parametrized Models (cont.)

Under arbitrary but infinitesimal perturbations however we have generically

$$\nu_1^+(\mathbf{a}) = 2, \quad \nu_2^+(\mathbf{b}) = 2, \quad \nu_3^+(\mathbf{c}) = 3, \quad \nu_4^+(\mathbf{d}) = 4.$$

These values are the values of ν_{max}^+ for each perturbation class respectively and in each case there is a drop in the value of ν^+ at the nominal.

Nominal and Parametrized Models (cont.)

Based on the above discussion we see that the McMillan degree

$$\nu(\mathbf{p}) = \nu[P(s, \mathbf{p})],$$

as well as $\nu^+(\mathbf{p})$, is a discontinuous function of $\mathbf{p}$ and in general its value drops on an algebraic variety. Indeed if $\mathcal{B}$ denotes an arbitrarily small ball in $\mathbb{R}^\ell$, centered at the origin we have:

$$\nu_{max} = \max_{\delta\mathbf{p}\in\mathcal{B}} \nu(\mathbf{p} + \delta\mathbf{p}).$$

Then the algebraic variety $\mathcal{V}$ is:

$$\mathcal{V} = \left\{\mathbf{p} \ : \ \nu(\mathbf{p}) \neq \nu_{max}\right\}.$$

Nominal and Parametrized Models (cont.)

In an exactly similar manner, we can write

$$\mathcal{V}^+ := \left\{ \mathbf{p} \; : \; \nu^+(\mathbf{p}) \neq \nu^+{}_{max} \right\}$$

to denote the algebraic variety where the McMillan degree of the antistable component drops. Referring to the previous example,we see, for example, that for $P_1(s, \mathbf{a})$:

$$\mathcal{V}_1^+ = \{\mathbf{a} : a_1 + a_2 + a_1 a_2 = 0\}.$$

Next, we discuss the effect of the discontinuity of $\nu^+(\mathbf{p})$ on feedback stabilization.

Structurally Stable Stabilization

A parametrized system is *structurally stable* if it remains stable under small but arbitrary perturbation of its parameters. Consider, in the previous example, a feedback controller C_0, of order q_0 that stabilizes the plant P_0 represented by the transfer function $P_0(s) = P(s, \mathbf{p}_0)$ in (1).

Equivalently C_0 internally stabilizes a 2^{nd} order realization of P_0. Now let us introduce small but arbitrary perturbations of $\mathbf{p}_0$. It is easy to see that

(a) if $P(s, \mathbf{p})$ is parametrized as in (2) such perturbations will introduce one unstable closed-loop pole close to $s = 1$.

(b) if $P(s, \mathbf{p})$ is parametrized as in (3) such perturbations will introduce one unstable closed-loop pole close to $s = 1$.

Structurally Stable Stabilization (cont.)

(c) if $P(s, \mathbf{p})$ is parametrized as in (4) such perturbations will introduce two unstable closed-loop poles close to $s = 1$.

(d) if $P(s, \mathbf{p})$ is parametrized as in (5) such perturbations will introduce three unstable closed-loop poles close to $s = 1$.

Therefore, in each case, the closed-loop is rendered unstable by infinitesimal arbitrary perturbations. Such a system is called *structurally* unstable.

Structurally Stable Stabilization (cont.)

To remedy the situation it is obviously necessary to know the perturbation class to which $P_0(s)$ belongs. This requires at least partial knowledge of internal structure and parametrization beyond the nominal transfer function. Once the correct parametrization is known we can perturb the parameter $\mathbf{p}_0$ to $\mathbf{p}_1$ in this class, so that

$$\nu^+ \left[P\left(s, \mathbf{p}_1\right)\right] = \nu^+_{max}$$

construct a minimal realization of order ν_{max} and stabilize it with a compensator C_1. Note that such a compensator generically stabilizes $P(s, \mathbf{p}_1)$ for small perturbations about $\mathbf{p}_1$. Thus, if $P(s, \mathbf{p})$ belongs to the parametrization $P_1(s, \mathbf{b})$ the controller must stabilize a 2^{nd} order plant, if it belongs to $P_2(s, \mathbf{c})$ it must stabilize a 3^{rd} order and if it belongs to $P_3(s, \mathbf{d})$ it must stabilize a 4^{th} order plant, respectively.

Structurally Stable Stabilization (cont.)

In each case, structurally stable stabilization will be achieved by nominally stabilizing a system of order ν_{max} for that class, rather than the minimal order realization of $P_0(s)$. In each case however the minimal realization of order ν_{max} of an arbitrary neighborhood of $P(s, \mathbf{p}_0)$ is destabilized by the compensator. These observations are summarized in the following:

Structurally Stable Stabilization (cont.)

Theorem

Assume that small perturbations in $\mathbf{p}_0$ *introduces some additional open RHP poles in* $P(s, \mathbf{p})$. *A plant with transfer function* $P(s, \mathbf{p}_0)$ *can be stabilized by a linear time-invariant feedback controller in a structurally stable manner iff*

$$\nu^+(\mathbf{p}_0) = \nu^+_{max}.$$

If $\nu^+(\mathbf{p}_0) \; < \; \nu^+_{max}$, *then*

- *any stabilizing controller for* $P(s, \mathbf{p}_0)$ *renders the closed loop is not structurally stable, that is, the closed loop is destabilized by arbitrarily small perturbations of the parameter* $\mathbf{p}_0$.
- *any controller that stabilizes* $P(s, \mathbf{p}_1)$ *with* $\nu^+(\mathbf{p}_1) = \nu^+_{max}$ *fails to stabilize a "ball" around* $\mathbf{p}_1$ *that includes plant* $P(s, \mathbf{p}_0)$.

Structurally Stable Stabilization (cont.)

In other words failure of the condition stated in the theorem implies that one must either give up structural stability or nominal stability as no controller can simultaneously achieve both.

Remark

It is important to note that it is possible to have a controller that simultaneously stabilizse the perturbed plant $P(\mathbf{p}_1)$ *of higher and the norminal plant* $P(s, \mathbf{p}_0)$ *of lower order. However, there is no controller that stabilizes the entire ball around* $\mathbf{p}_1$ *that includes the nominal* $\mathbf{p}_0$*. This means that some plant of higher order along any path* $(\mathbf{p}_1, \mathbf{p}_0)$ *will be destabilized even though the controller simultaneously stabilizes* $P(\mathbf{p}_1)$ *and* $P(\mathbf{p}_0)$*.*

Structurally Stable Stabilization (cont.)

Remark

In this theorem, we assume that the balls of plants being considered are all of degree ν_{max}.

Remark

On any continuous path $\mathbf{p}_\lambda$ *connecting* $\mathbf{p}_1$ *and* $\mathbf{p}_0$, *there will be a point* λ^* *such that system of order* $\nu_{max} + \text{order of controller}$ *with realization of the transfer function* $P(s, \lambda^*)$ *of order* ν_{max} *will have* $j\omega$ *eigenvalues. These may or may not correspond to uncontrollable/unobservable eigenvalues. This is determined by thether the path intersects the algebraic variety* $\mathcal{V}^+$.

Structurally Stable Stabilization (cont.)

The proof is a formalization of the above remarks and is best stated utilizing the following Lemma. We say an $n \times n$ matrix is stable (unstable) if it contains all (some) eigenvalues in the open left half plane (closed right half plane).

Lemma

Let $A(\mathbf{p})$ denote an $n \times n$ real or complex matrix whose entries are continuous functions of the real parameter vector $\mathbf{p} \in \mathbb{R}^l$. If $A(\mathbf{p}_0)$ is stable, then $A(\mathbf{p}_0 + \epsilon)$ is stable, for sufficiently small ϵ, in a ball $\mathcal{B} \in \mathbb{R}^l$. Similarly, If $A(\mathbf{p}_0)$ is unstable with at least one pole at the open RHP, then $A(\mathbf{p}_0 + \epsilon)$ is unstable with at least one pole at the open RHP, for sufficiently small ϵ, in a ball $\mathcal{B} \in \mathbb{R}^l$.

The proof of the Lemma is a straightforward consequence of the facts that the characteristic polynomial of A is degree invariant and the eigenvalues of A are continuous functions of its entries. The proof of the Theorem can now be stated:

Structurally Stable Stabilization (cont.)

Proof of Theorem

Let C_0 be a controller, say of order q_0, internally stabilizing a stabilizable and detectable realization of $P(s, \mathbf{p}_0)$, say of order n_0. Denote the $n_0 + q_0$ closed loop eigenvalues by Λ_0. To discuss structural stability, we fix C_0, perturb $\mathbf{p}_0$ to $\mathbf{p}_1 = \mathbf{p}_0 + \epsilon$, and replace the plant by a stabilizable and detectable realization of $P(s, \mathbf{p}_1)$, say of order n_1. Now suppose $\nu^+(\mathbf{p}_0) = \nu^+_{max}$. Then for ϵ sufficiently small but arbitrary, $\nu^+(\mathbf{p}_1) = \nu^+(\mathbf{p}_0)$ and we may take $n_1 = n_0$. By the Lemma above the closed loop eigenvalues with the plant $P(s, \mathbf{p}_1)$ and controller C_0 remains close to those of Λ_0 and therefore in the LHP.

Structurally Stable Stabilization (cont.)

On the other hand suppose that $\nu^+(\mathbf{p}_0) < \nu^+_{max}$. Thus, $\nu^+(\mathbf{p}_1) = \nu^+_{max} > \nu^+(\mathbf{p}_0)$. Now let n_1 denote any stabilizable and detectable realization of $P(s, \mathbf{p}_1)$ and consider the $n_1 + q_0$ closed loop eigenvalues, denoted $\Lambda_1(\mathbf{p}_0 + \epsilon)$ of this plant with the fixed controller C_0. In this case it is impossible to take $n_1 = n_0$.

Moreover as $\epsilon \to 0$ any state space realization of $P(s, \mathbf{p}_1)$ contains $\nu^+(\mathbf{p}_1) - \nu^+(\mathbf{p}_0)$ uncontrollable and/or unobservable RHP eigenvalues and are therefore contained in $\Lambda_1(\mathbf{p}_0)$. By the Lemma above, we conclude that closed loop is unstable for sufficiently small, but arbitrary ϵ.

Structurally Stable Stabilization (cont.)

Example

Consider the plant with transfer function parametrization:

$$y(s) = P(s, \delta)u(s)$$

where

$$P(s, \delta) = \begin{bmatrix} \dfrac{2}{(s+1)(s-1)} & \dfrac{1+\delta}{s-1} \\ \dfrac{1}{s-1} & \dfrac{1}{s-1} \end{bmatrix}. \qquad (6)$$

With $\delta = 0$,

$$P(s, 0) =: P_0(s) = \begin{bmatrix} \dfrac{2}{(s+1)(s-1)} & \dfrac{1}{s-1} \\ \dfrac{1}{s-1} & \dfrac{1}{s-1} \end{bmatrix}. \qquad (7)$$

Structurally Stable Stabilization (cont.)

We consider the stabilizing controller

$$u = -Ky + v \tag{8}$$

with

$$K = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

A minimal realization of $P_0(s)$ is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} x \end{aligned} \tag{9}$$

and the closed loop system is

$$\dot{x} = (A - BKC)x + Bv \tag{10}$$

Structurally Stable Stabilization (cont.)

is internally stable with the controller (17) with characteristic polynomial

$$s^2 + 9s + 12.$$

Now consider a "small" perturbation of $P_0(s)$ obtained by letting δ be nonzero. A minimal realization of (6) with $\delta \neq 0$ is:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1+\delta \\ 1 & 1 \end{bmatrix} u \qquad (11) \\ y &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \end{aligned}$$

and the closed loop system with the previous controller is

$$\dot{x} = \begin{bmatrix} 0 & -1 & -2 \\ 4+3\delta & -3-3\delta & -6-4\delta \\ 4 & -4 & -5 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1+\delta \\ 1 & 1 \end{bmatrix} v. \qquad (12)$$

Structurally Stable Stabilization (cont.)

The characteristic polynomial of (12) is

$$s^3 + (8 + 3\delta)s^2 + (3 + 2\delta)s - \delta - 12 \tag{13}$$

and is seen to be unstable for "small" values of δ, and in this particular case for all values of δ. Moreover, as $\delta \to 0$, a root to (13) tends to $s = 1$.

A remedy to the structural instability above would be to design a stabilizing controller for the third order model (11), say, for some $\delta = \delta^0$. Such a controller would stabilize a ball of plants around δ^0 but cannot obviously stabilize the plant with $\delta = 0$.

Structurally Stable Stabilization (cont.)

Example

Consider the plant with transfer function

$$P(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s-1} \\ \frac{1}{s-1} & \frac{1}{s-1} \end{bmatrix}. \qquad (14)$$

Its minimal realization is

$$\dot{x} = [1]x + [1 \quad 1]u, \qquad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x. \qquad (15)$$

Now consider a perturbed plant

$$P(s,\delta) = \begin{bmatrix} \frac{1}{s-1} & \frac{1+\delta}{s-1} \\ \frac{1}{s-1} & \frac{1}{s-1} \end{bmatrix} \qquad (16)$$

Structurally Stable Stabilization (cont.)

Its minimal realization is

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 1+\delta \\ 1 & 1 \end{bmatrix} x \qquad (17)$$

For convenience let $\mathbf{p}_1 = \delta > 0$ (real), then the following controller stabilizes the system.

$$K = \begin{bmatrix} 1 & -3 \\ -k_1 & k_1 \end{bmatrix}. \qquad (18)$$

The closed-loop characteristic polynomial of this sytem is

$$\Pi_1(s) = (s+1)(s+k_1\delta-1). \qquad (19)$$

It shows that for any given perturbation δ, there is a controller value that robustly stabilizes the entire family in $\mathbf{p} \in (0, \delta]$. Furthermore, the order of a minimal realization of every plant in the family is of 2. However, when $\delta = 0$, the controller should be

Structurally Stable Stabilization (cont.)

applied to its minimal realization which is of order 1. In this case, the closed-loop characteristic polynomial becomes

$$\Pi_0(s) = s + 1 \tag{20}$$

and shows that the controller stabilizes the nominal plant of lower order. However, the closed-loop system is unstable for $\delta = \frac{1}{k_1}$ (i.e., higher order systems). In other words, for any given perturbation δ^*, no controller can stabilize the entire family of the plant for all $\delta \in (0, \delta^*)$ if and only if $\nu^+(\mathbf{p}_0) \neq \nu^+(\mathbf{p})$.