ECEN 605

LINEAR SYSTEMS

Lecture 20

Characteristics of Feedback Control Systems II
– Feedback and Stability

Feedback System

Consider the feedback system

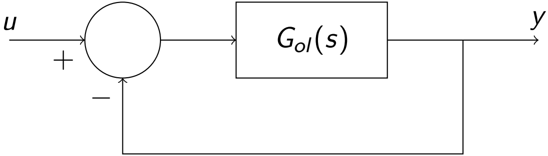

Figure 1: A unity feedback system

where $G_{ol}(s)$ is the **open loop** transfer function.

Feedback System (cont.)

The **closed loop** transfer function is

$$\frac{Y(s)}{U(s)} = \frac{G_{ol}(s)}{1 + G_{ol}(s)} =: G_{cl}(s). \tag{1}$$

If the open loop transfer function is written in terms of numerator and denominator polynomials,

$$G_{ol}(s) = \frac{n_{ol}(s)}{d_{ol}(s)} \tag{2}$$

we have

$$G_{cl}(s) = \frac{n_{cl}(s)}{d_{cl}(s)} = \frac{n_{ol}(s)}{d_{ol}(s) + n_{ol}(s)}, \tag{3}$$

Feedback System (cont.)

so that

$$n_{cl}(s) = n_{ol}(s) \tag{4}$$

and

$$d_{cl}(s) = d_{ol}(s) + n_{ol}(s). \tag{5}$$

Equations (4) and (5) show that

a) the zeros of the unity feedback system are identical to the zeros of the open loop system,

b) the poles of the closed loop system in general differ from those of the open loop system and are given by the roots of $d_{cl}(s) = 0$.

Feedback System (cont.)

In view of the above

$$d_{cl}(s) = d_{ol}(s) + n_{ol}(s) \tag{6}$$

is called the **closed loop characteristic polynomial**. It follows that the closed loop system is stable if and only if all roots of

$$d_{cl}(s) = 0 \tag{7}$$

called **closed loop characteristic roots** lie in the open left half of the complex plane (LHP), or equivalently $d_{cl}(s)$ is a **Hurwitz polynomial**.

Feedback System (cont.)

Example

$$G_{ol}(s) = \frac{K(s-1)}{(s+1)} \tag{8}$$

$$G_{cl}(s) = \frac{K(s-2)}{(s+1)+K(s-2)} \tag{9a}$$

$$= \frac{K(s-2)}{(1+K)s+(1-2K)} \tag{9b}$$

Feedback System (cont.)

If $K = 1$,

$$d_{cl}(s) = 2s - 1 = 0 \quad \Rightarrow \quad s = {}^1\!/_2, \tag{10}$$

but if $K = {}^1\!/_4$

$$d_{cl}(s) = {}^5\!/_4\, s + {}^1\!/_2 = 0 \quad \Rightarrow \quad s = -{}^2\!/_5. \tag{11}$$

This example shows that the closed loop may be stable or unstable, depending on the value of K, even though the open loop is stable.

Feedback System (cont.)

Example

$$G_{ol}(s) = \frac{K}{s-2} \tag{12}$$

$$G_{cl}(s) = \frac{K}{s-2+K} \tag{13}$$

Thus the open loop system is unstable but the feedback system is stable for $K > 2$.

Control System

In a control system containing plant and controller in unity feedback we have

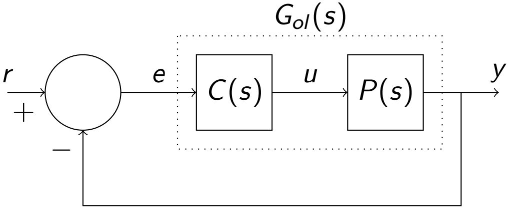

Figure 2: Unity feedback system with a controller and a plant

Control System (cont.)

$$G_{ol}(s) = P(s)\,C(s) \tag{14}$$

$$G_{cl}(s) = \frac{P(s)\,C(s)}{1 + P(s)\,C(s)}. \tag{15}$$

Writing the transfer functions in terms of numerator and denominator polynomials

$$G_{ol}(s) = \underbrace{\frac{n_P(s)}{d_P(s)}}_{P(s)}\;\underbrace{\frac{n_C(s)}{d_C(s)}}_{C(s)} \tag{16}$$

$$G_{cl}(s) = \frac{n_P(s)\,n_C(s)}{d_P(s)\,d_C(s) + n_P(s)\,n_C(s)}. \tag{17}$$

Control System (cont.)

Therefore the closed loop characteristic polynomial is given by

$$d_{cl}(s) = d_P(s)\, d_C(s) + n_P(s)\, n_C(s). \tag{18}$$

The roots of $d_{cl}(s) = 0$ are the closed loop characteristic roots and a controller $C(s)$ **stabilizes** a plant $P(s)$ if and only if the roots lie in the LHP, or $d_{cl}(s)$ is Hurwitz.

Control System (cont.)

Example

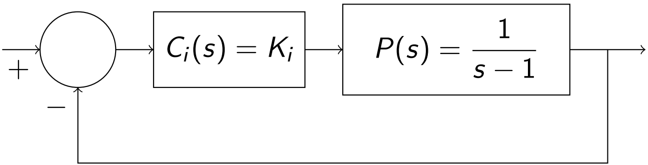

Figure 3: A gain controller with an unstable plant

Control System (cont.)

The controller $C_1(s) = K_1 = 2$ stabilizes the closed loop system since $d_{cl}(s) = s - 1 + K_1 = s + 1$ has an LHP root at $s = -1$. The controller $C_2(s) = K_2 = 1/2$ does not stabilize the closed loop system since $d_{cl}(s) = s - 1 + K_2 = s - 1/2$ has an RHP root at $s = 1/2$.

Control System (cont.)

Example (Routh Criterion)

Consider the following system.

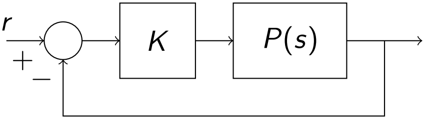

Figure 4: A gain controller K with plant $P(s)$

Control System (cont.)

Find the range of stabilizing K.

$a)\ P(s) = \dfrac{s-1}{s(s+1)}$

$b)\ P(s) = \dfrac{1}{s(s+1)(s+2)}$

$c)\ P(s) = \dfrac{1}{s(s+1)(s+2)(s+3)}$

$d)\ P(s) = \dfrac{s+1}{s^2(s+2)}$

$e)\ P(s) = \dfrac{s+2}{s^2(s+1)}$

$f)\ P(s) = \dfrac{s-1}{(s+1)(s-2)}$

Control System (cont.)

Solution. In each case apply the Routh criterion to the characteristic polynomial of the closed loop system:

a)

$$\begin{aligned} d_{cl}(s) &= s(s+1) + K(s-1) \\ &= s^2 + (K+1)s - K. \end{aligned} \tag{19}$$

Stabilizing range : $-1 < K < 0$.

b)

$$\begin{aligned} d_{cl}(s) &= s(s+1)(s+2) + K \\ &= s^3 + 3\,s^2 + 2\,s + K. \end{aligned} \tag{20}$$

Stabilizing range : $0 < K < 6$.

Control System (cont.)

$c)$

$$d_{cl}(s) = s(s+1)(s+2)(s+3) + K = s^4 + 6\,s^3 + 11\,s^2 + 6\,s + K. \quad (21)$$

Stabilizing range : $0 < K < 10.$

$d)$

$$d_{cl}(s) = s^2(s+2) + K(s+1) = s^3 + 2\,s^2 + K\,s + K. \quad (22)$$

Stabilizing range : $0 < K.$

$e)$

$$d_{cl}(s) = s^2(s+1) + K(s+2) = s^3 + s^2 + K\,s + 2K. \quad (23)$$

Stabilizing range : *does not exist*.

Tracking Step Inputs

In this section we discuss the problem of designing a controller to track step inputs. Consider first the system

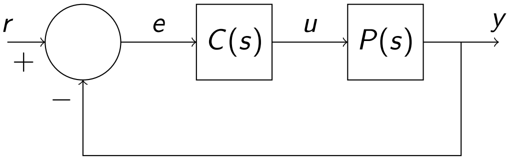

Figure 5: A unity feedback system

where

$$P(s) = \frac{n_P(s)}{d_P(s)}, \qquad C(s) = \frac{n_C(s)}{d_C(s)}. \tag{24}$$

Tracking Step Inputs (cont.)

The error transfer function is given by

$$E(s) = \frac{1}{1 + P(s)\,C(s)} R(s) \tag{25a}$$

$$= \frac{d_P(s)\,d_C(s)}{d_P(s)\,d_C(s) + n_P(s)\,n_C(s)} R(s) \tag{25b}$$

$$= \frac{d_P(s)\,d_C(s)}{d_{cl}(s)} R(s). \tag{25c}$$

Tracking Step Inputs (cont.)

Suppose now that $r(t)$ is a step input of height R_0

$$R(s) = \frac{R_0}{s} \tag{26}$$

and

$$E(s) = \frac{d_P(s)\, d_C(s)}{d_{cl}(s)} \frac{R_0}{s}. \tag{27}$$

Thus the forced response $e(t) = \mathcal{L}^{-1}(E(s))$ consists of weighted exponential terms $e^{\lambda_i t}$, λ_i being the closed loop characteristic roots and a constant term E_{ss} which is the **steady state error**.

Tracking Step Inputs (cont.)

Since the closed loop must be stable the λ_i must have negative real parts and so E_{ss} is the residue of the pole at $s = 0$ in a partial fraction expansion of $E(s)$. Thus, by usual partial fraction formula,

$$\lim_{t \to \infty} e(t) = E_{ss} = \frac{d_P(0)\, d_C(0)\, R_0}{d_{cl}(0)} \tag{28}$$

is the steady state error.

Tracking Step Inputs (cont.)

Example

Consider the plant $P(s) = \dfrac{1}{s-2}$ and controller $C(s) = K$. As seen before the closed loop is stable if and only if $K > 2$. For a unit step input the steady state error is

$$E = \frac{-2 \cdot 1}{-2 + K} = \frac{-2}{K-2} \tag{29}$$

Tracking Step Inputs (cont.)

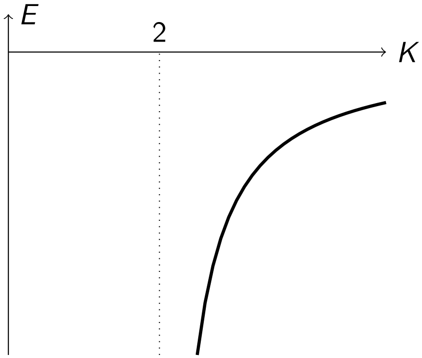

Figure 6:

which shows that the steady state error can be reduced to zero only with infinite gain.

Tracking Step Inputs (cont.)

Example

Consider the system

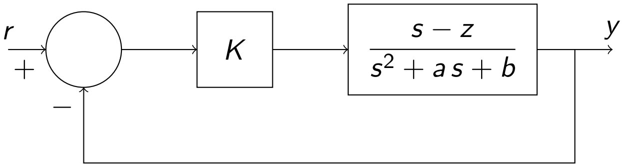

Figure 7: A gain controller

subject to a unit step input $r(t)$.

Tracking Step Inputs (cont.)

The closed loop characteristic polynomial

$$d_{cl}(s) = (s^2 + a\,s + b) + K(s - z) \tag{30a}$$

$$= s^2 + (a + K)s + (b - K\,z) \tag{30b}$$

and closed loop stability is equivalent to

$$K + a > 0 \tag{31a}$$

$$b - K\,z > 0 \tag{31b}$$

so that

$$-a < K < \frac{b}{z}. \tag{32}$$

Tracking Step Inputs (cont.)

The steady state error to a unit step is

$$E_{ss} = \frac{d_C(0)\, d_P(0)}{d_{cl}(0)} = \frac{1 \cdot b}{b - K\, z}. \tag{33}$$

Assuming that the range in (32) is nonempty, that is

$$\frac{b}{z} > -a \tag{34}$$

we can sketch the function (33).

Tracking Step Inputs (cont.)

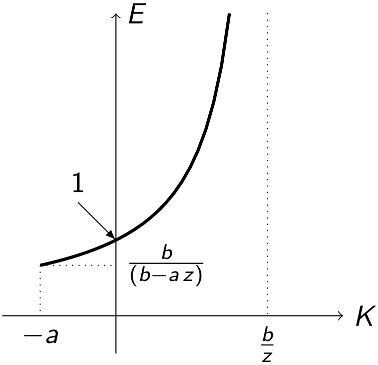

Figure 8:

We note that in this example the steady state error has an infimum or a lower bound of $\dfrac{b}{b - a\,z}$, over the stabilizing range of controller gains.