# ECEN 605

## LINEAR SYSTEMS

Lecture 21

Characteristics of Feedback Control Systems III
– Integral Control and Pole Placement Compensator

# Integral Control

In the following example we show that including an integrator in the loop driven by the tracking can allow us to achieve zero steady state tracking error.

# Integral Control (cont.)

## Example

Consider the plant of the previous example with the integral controller $C(s) = \dfrac{K}{s}$,

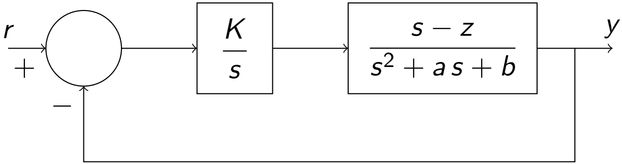


Figure 1: A feedback loop with an integral controller

# Integral Control (cont.)

We see that

$$d_{cl}(s) = s^3 + a\,s^2 + (b + K)s - Kz \tag{1}$$

and using the Routh criterion the stability conditions are

$$\begin{aligned} a &> 0 && \text{(2a)} \\ b + K &> 0 && \text{(2b)} \\ -K\,z &> 0 && \text{(2c)} \\ a(b + K) &> -K\,z. && \text{(2d)} \end{aligned}$$

For example, let $a = 1$, $b = 1$, $z = -1$, then (2) is satisfied if

$$K > 0 \qquad \text{(Stability condition).} \tag{3}$$

# Integral Control (cont.)

The steady state error to a unit step is **zero** since

$$E_{ss} = \frac{d_C(0)\, d_P(0)}{d_{cl}(0)} = \frac{0 \cdot b}{-K\, z} = 0 \tag{4}$$

for all $K$ satisfying the stability condition (2). We see that this holds due to the presence of the integrator in the controller.

# Integral Control (cont.)

In this example the stability condition (2a) is independent of $K$. If $a$ is negative or zero, stability cannot be attained by adjusting $K$.

A possible solution to this is a more general controller such as the proportional-integral (PI) controller

$$C(s) = K_P + \frac{K_I}{s} = \frac{s\,K_P + K_I}{s}. \tag{5}$$

With the controller (5) the closed loop characteristic polynomial is

$$\begin{aligned} d_{cl}(s) &= s(s^2 + a\,s + b) + (s\,K_P + K_I)(s - z) && \text{(6a)} \\ &= s^3 + (a + K_P)s^2 + (b - z\,K_P + K_I)s - z\,K_I. && \text{(6b)} \end{aligned}$$

# Integral Control (cont.)

Thus the stability conditions, from the Routh criterion, are:

$$a + K_P > 0 \tag{7a}$$

$$b - z\,K_P + K_I > 0 \tag{7b}$$

$$-K_I\,z > 0 \tag{7c}$$

$$(a + K_P)(b - z\,K_P + K_I) + K_I\,z > 0. \tag{7d}$$

and the steady state error to a unit step is

$$E_{ss} = \frac{d_C(0)\,d_P(0)}{d_{cl}(0)} = \frac{0 \cdot b}{-z\,K_I} = 0. \tag{8}$$

We note that **stability cannot be achieved if** $z = 0$, that is, the plant has a zero at the origin.

# Integral Control (cont.)

### Example

Consider the feedback control of the unstable plant, with transfer function

$$P(s) = \frac{1}{s-1}. \qquad (9)$$

# Integral Control (cont.)

We construct the feedback loop

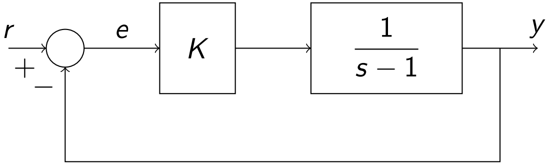


Figure 2: A feedback loop with an unstable plant

and first attempt to determine the range of controller gain $K$, that renders the closed loop system stable.

# Integral Control (cont.)

The closed loop characteristic polynomial is:

$$d_{cl}(s) = (s - 1) + K \tag{10}$$

and so the closed loop system is stable if

$$K > 1. \tag{11}$$

# Integral Control (cont.)

As shown in the diagram below the gain $K$ "pushes" the open loop pole at $s = 1$ to the left and succeeds in pushing it into the left half plane only when $K > 1$.

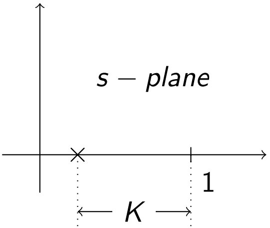


Figure 3: The location of the pole depending on the value of $K$

# Integral Control (cont.)

Now, let us consider the task of tracking a unit step in $r(t)$, the reference signal. The Laplace transform $E(s)$ of the error $e(t)$ is given by:

$$E(s) = \frac{s-1}{(s-1+K)}\frac{1}{s} \tag{12}$$

and so assuming $K$ satisfies (11)

$$\begin{aligned} E_{ss} &= \lim_{s\to 0} s\,E(s) \\ &= \frac{-1}{-1+K} = \frac{-1}{K-1}. \end{aligned} \tag{13}$$

# Integral Control (cont.)

The graphical representation of (13) is shown below:

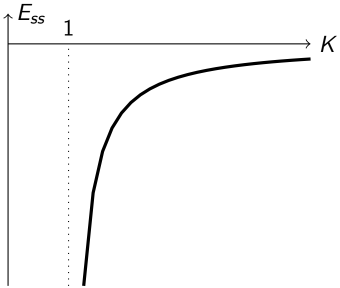


Figure 4:

It is seen that $|E_{ss}| \to 0$ only when $K \to \infty$.

# Integral Control (cont.)

To obtain zero steady state error without infinite gain we know that integral control is necessary. If

$$C(s) = \frac{K_C}{s} \tag{14}$$

is a proposed controller the closed loop system is:

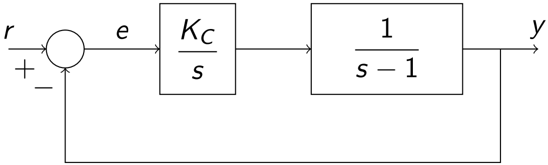


Figure 5: An integral controller with the unstable plant

# Integral Control (cont.)

The closed loop characteristic polynomial for this system is:

$$d_{cl}(s) = s^2 - s + K_C \tag{15}$$

and this clearly shows that $K_c$ is unable to stabilize the closed loop system.

# Integral Control (cont.)

With the above analysis in hand consider the Proportional Integral (PI) controller, with transfer function

$$C(s) = K_P + \frac{K_I}{s} = \frac{s\,K_P + K_I}{s}. \tag{16}$$

The closed loop system is

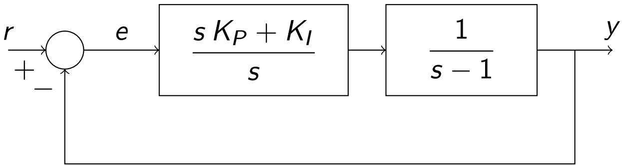


Figure 6: A PI controller with the unstable plant

# Integral Control (cont.)

and the closed loop characteristic polynomial is

$$\begin{aligned} d_{cl}(s) &= s(s-1) + s\,K_P + K_I \\ &= s^2 + s(K_P - 1) + K_I. \end{aligned} \tag{17}$$

# Integral Control (cont.)

Clearly, $K_P$ and $K_I$ can be chosen to arbitrarily assign closed loop characteristic roots to (17) and in particular render $d_{cl}(s)$ Hurwitz and thus the closed loop system stable. The stabilizing region is described in the $K_P$, $K_I$ plane by

$$K_P > 1, \quad K_I > 0 \tag{18}$$

and is displayed below:

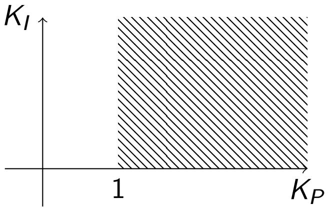


Figure 7: The stabilizing region

# Integral Control (cont.)

Note that the closed loop transfer functions are:

$$Y(s) = \frac{s\,K_P + K_I}{s^2 + (K_P - 1)s + K_I} R(s) \tag{19}$$

and

$$E(s) = \frac{s(s-1)}{s^2 + (K_P - 1)s + K_I} R(s). \tag{20}$$

The steady state error to a unit ramp ($u(t) = t$) is

$$\begin{aligned} E(s) &= \frac{d_C(s)\,d_P(s)}{d_{cl}(s)} R(s) \\ &= \frac{s(s-1)}{s^2 + s(K_P - 1) + K_I} \frac{1}{s^2} \\ &= \frac{s-1}{s^2 + s(K_P - 1) + K_I} \frac{1}{s} \\ &= \frac{A_0}{s} + \frac{A_1}{s - \lambda_1} + \frac{A_2}{s - \lambda_2}. \end{aligned} \tag{21}$$

# Integral Control (cont.)

So,

$$E_{ss} = A_0 = \frac{-1}{K_I}. \tag{22}$$

If we want the magnitude of the error to be less than 0.05, then

$$\left| -\frac{1}{K_I} \right| < 0.05. \tag{23}$$

Thus,

$$|K_I| > 20. \tag{24}$$

# Integral Control (cont.)

## Example

Consider the control system,

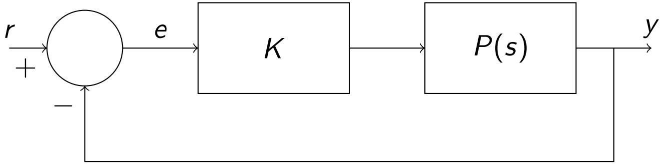


Figure 8: A gain controller

with,

# Integral Control (cont.)

a) $P(s) = \dfrac{s+1}{s(s-1)(s+6)}$
The closed loop characteristic polynomial is

$$\begin{aligned} d_{cl}(s) &= s(s-1)(s+6) + K(s+1) \\ &= s^3 + 5\,s^2 + (K-6)s + K. \end{aligned} \tag{25}$$

For the system to be stable,

$$K - 6 > 0, \tag{26}$$
$$K > 0, \tag{27}$$
$$5(K-6) > K. \tag{28}$$

Hence, any $K > 7.5$ will stabilize the system.

# Integral Control (cont.)

b) $P(s) = \dfrac{1}{s(s+1)(s+2)}$
The closed loop characteristic polynomial is

$$\begin{aligned} d_{cl}(s) &= s(s+1)(s+2) + K \\ &= s^3 + 3\,s^2 + 2s + K. \end{aligned} \tag{29}$$

For the system to be stable,

$$K > 0, \tag{30}$$

$$6 > K. \tag{31}$$

Hence any $K$ such that $0 < K < 6$ will stabilize the system.

# Integral Control (cont.)

Moreover, for the steady state error to a unit ramp, the error transfer function is

$$E(s) = \frac{d_C(s)\, d_P(s)}{d_{cl}(s)} \frac{1}{s^2} = \frac{s(s+1)(s+2)}{s^3 + 3\, s^2 + 2s + K} \frac{1}{s^2} = \frac{2/K}{s} + \frac{A_1}{s-\lambda_1} + \frac{A_2}{s-\lambda_2} + \frac{A_3}{s-\lambda_3}. \tag{32}$$

Therefore the steady state error, denoted as $E_{ss,\, ramp}$, is

$$E_{ss,\, ramp} = \frac{2}{K}. \tag{33}$$

Note that by (31), the minimum error bound is

$$E_{ss,\, ramp} > \frac{1}{3}. \tag{34}$$

# Pole Placement Compensator

## Example

Consider the system,

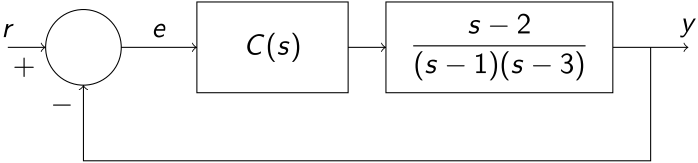


Figure 9: A pole placement compensator

We are to design $C(s)$ so that $y$ tracks a step input $r$ with zero steady state error and choose closed loop poles so that the error goes to zero in 5 seconds.

# Pole Placement Compensator (cont.)

The controller must have the form,

$$C(s) = \frac{\alpha_0 + \alpha_1 s + \alpha_2 s^2}{s(s + \beta_0)} \tag{35}$$

in order to have sufficient freedom to assign all closed loop poles arbitrarily and "push" them to the left of the line $\mathbf{Re}\, s = -1$, to achieve error convergence in 5 *secs*, corresponding to a time constant of 1 *sec*.

The closed loop characteristic polynomial is

$$d_{cl}(s) = s(s + \beta_0)(s - 1)(s - 3) + (\alpha_0 + \alpha_1 s + \alpha_2 s^2)(s - 2). \tag{36}$$

# Pole Placement Compensator (cont.)

The time constant of the system have to be $\leq 1$. Choosing poles at $-1 \pm j$, $-2$, $-2$ we have

$$\begin{aligned} \left[(s+1)^2+1\right][s+2]^2 &= (s^2+2\,s+2)(s^2+4\,s+4) \\ &= s^4+6\,s^3+14\,s^2+16\,s+8. \end{aligned} \tag{37}$$

Equation (36) can be expanded as

$$\begin{aligned} & (s^2+\beta_0\,s)(s^2-4\,s+3) \\ & +\alpha_0\,s+\alpha_1\,s^2+\alpha_2\,s^3-2\,\alpha_0-2\,\alpha_1\,s-2\,\alpha_2\,s^2 \\ = \quad & s^4+s^3\,(\beta_0-4+\alpha_2)+s^2\,(3-4\,\beta_0+\alpha_1-2\,\alpha_2) \\ & +s\,(3\,\beta_0+\alpha_0-2\,\alpha_1)-2\,\alpha_0. \end{aligned} \tag{38}$$

# Pole Placement Compensator (cont.)

Equating coefficients in (37) and (38)

$$-2\,\alpha_0 = 8 \quad (\alpha_0 = -4) \tag{39}$$

$$3\,\beta_0 + \alpha_0 - 2\,\alpha_1 = 16 \quad (30 - 3\,\alpha_2 - 4 - 2\,\alpha_1 = 16) \tag{40}$$

$$3 - 4\,\beta_0 + \alpha_1 - 2\,\alpha_2 = 14 \quad (3 - 40 + 4\,\alpha_2 + \alpha_1 - 2\,\alpha_2 = 14) \tag{41}$$

$$\beta_0 - 4 + \alpha_2 = 6 \quad (\beta_0 = 10 - \alpha_2). \tag{42}$$

These reduce to

$$\alpha_1 + 2\,\alpha_2 = 51 \tag{43}$$

$$2\,\alpha_1 + 3\,\alpha_2 = 10 \tag{44}$$

so that $\alpha_2 = 92$, $\alpha_1 = -133$, $\beta_0 = -82$, $\alpha_0 = -4$.

# Pole Placement Compensator (cont.)

Therefore the controller transfer function is

$$C(s) = \frac{-4 - 133\,s + 92\,s^2}{s(s-82)}. \tag{45}$$

In addition, we find the steady state error to a unit ramp. The Laplace transform of the error is

$$\begin{aligned} E(s) &= \frac{d_C(s)\,d_P(s)}{d_{cl}(s)}\frac{1}{s^2} \\ &= \frac{s(s-82)}{s^4 + 6\,s^3 + 14\,s^2 + 16\,s + 8}\frac{1}{s^2}. \end{aligned} \tag{46}$$

Hence, the steady state error to a unit ramp is $\dfrac{-82}{8}$.