ECEN 605

LINEAR SYSTEMS

Lecture 23

Characteristics of Feedback Control Systems IV
– Linear Servomechanisms

Linear Servomechanisms

In this section we extend the previous results on tracking and rejection of step inputs to more general classes of persistent or unstable signals. For example the reference signal may be arbitrary steps and ramps and the disturbance may be steps and/or sinusoidal signals of known frequency ω_0, and arbitrary amplitude and phase. In this case,

$$D^2 \, r(t) = 0 \tag{1a}$$

$$D \, (D^2 + \omega_0^2) \, d(t) = 0 \tag{1b}$$

where $D \equiv \frac{d}{d\,t}$ is the differentiation operator.

Linear Servomechanisms (cont.)

Taking Laplace transforms we see that

$$R(s) = \frac{r_0 + r_1\, s}{s^2} = \frac{n_R(s)}{s^2} \tag{2}$$

and

$$D(s) = \frac{d_0 + d_1\, s + d_2\, s^2}{s(s^2 + \omega_0^2)} = \frac{n_D(s)}{s(s^2 + \omega_0^2)}. \tag{3}$$

Linear Servomechanisms (cont.)

As r_0, r_1 vary (2) generates all steps and ramps in $r(t)$ and as d_0, d_1, d_2 vary (3) generates all steps and sinusoids of frequency ω_0 in $d(t)$. The polynomial

$$m(s) = s^2(s^2 + \omega_0^2) \tag{4}$$

is the lowest degree annihilating polynomial for $r(t)$ and $d(t)$ in the sense that

$$m(D)\, r(t) = 0 \tag{5a}$$
$$m(D)\, d(t) = 0 \tag{5b}$$

for all $r(t)$, $d(t)$ satisfying (1).

Linear Servomechanisms (cont.)

More generally if (1a) and (1b) are replaced by

$$m_r(D)\, r(t) = 0 \tag{6a}$$

$$m_d(D)\, d(t) = 0 \tag{6b}$$

then

$$m(D) = l\,c\,m\,(m_r(D), m_d(D)) \tag{7}$$

is the lowest degree polynomial such that

$$m(D)\, r(t) = 0 \tag{8a}$$

$$m(D)\, d(t) = 0 \tag{8b}$$

for all $(r(t),\ d(t))$ satisfying (6) where $l\,c\,m$ denotes the least common multiple.

Linear Servomechanisms (cont.)

As in (6) we then have

$$R(s) = \frac{n_R(s)}{m_r(s)} \tag{9}$$

$$D(s) = \frac{n_D(s)}{m_d(s)}. \tag{10}$$

In this section we assume that the signals to be tracked or rejected are persistent (steps, sinusoids, etc) or unstable (ramps, exponentially increasing, etc) since a stable feedback loop asymptotically "tracks" signals converging to zero anyway, and is the "uninteresting case."

Linear Servomechanisms (cont.)

Now, let us reconsider the control problem

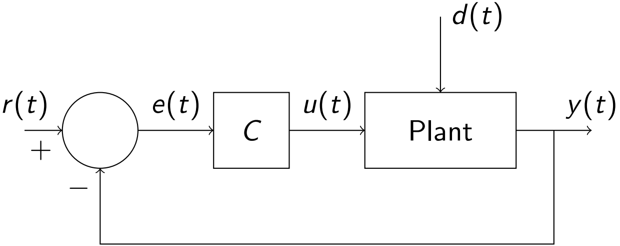

Figure 1: A closed loop system with disturbance

where the controller C is to be designed to track any reference signal $r(t)$ generated by (6a) and reject any disturbance $d(t)$ generated by (6b).

Linear Servomechanisms (cont.)

Writing the signals and controller and plant equations in the transfer function domain we have:

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ U(s) &= C(s)\,E(s) \\ Y(s) &= P(s)\,U(s) + Q(s)\,D(s). \end{aligned} \tag{11}$$

Linear Servomechanisms (cont.)

With

$$\begin{aligned} P(s) &= \frac{n_P(s)}{d_P(s)} \\ Q(s) &= \frac{n_Q(s)}{d_Q(s)} \\ C(s) &= \frac{n_C(s)}{d_C(s)} \end{aligned} \tag{12}$$

we have

$$E(s) = \frac{d_P(s)\, d_C(s)}{d_{cl}(s)} R(s) - \frac{n_Q(s)\, d_C(s)}{d_{cl}(s)} D(s). \tag{13}$$

Linear Servomechanisms (cont.)

Finally using (6), that is

$$R(s) = \frac{n_R(s)}{m_r(s)}, \quad D(s) = \frac{n_D(s)}{m_d(s)} \tag{14}$$

we have

$$E(s) = \underbrace{\frac{d_P(s)\, d_C(s)}{d_{cl}(s)} \frac{n_R(s)}{m_r(s)}}_{E_r(s)} - \underbrace{\frac{n_Q(s)\, d_C(s)}{d_{cl}(s)} \frac{n_D(s)}{m_d(s)}}_{E_d(s)} \tag{15}$$

where, $m_r(s)$ and $m_d(s)$ are the characteristic polynomials of $R(s)$ and $D(s)$, respectively, and

$$d_{cl}(s) = d_C(s)\, d_P(s) + n_C(s)\, n_P(s). \tag{16}$$

Linear Servomechanisms (cont.)

For closed loop stability all the closed loop characteristic roots represented by the roots of $d_{cl}(s) = 0$ must lie in the open LHP.

This guarantees that zero input response for $e(t) \to 0$ as $t \to \infty$.

The zero state component or forced response of $e(t)$ is given by the inverse Laplace transform of $E(s)$ given by (15).

Linear Servomechanisms (cont.)

Since this must happen for arbitrary $n_R(s)$ and $n_D(s)$, that is, for all possible signals satisfying (6), we see that the poles of $E_r(s)$ and $E_d(s)$ must be stable implying that

$$m_r(s) \,|\, (d_P(s)\, d_C(s)) \tag{17a}$$

$$m_d(s) \,|\, (n_Q(s)\, d_C(s)). \tag{17b}$$

Here, $A(s)|B(s)$ means $A(s)$ divides $B(s)$, i.e., $\dfrac{B(s)}{A(s)}$ is a polynomial.

Linear Servomechanisms (cont.)

To guarantee that (17) holds for an arbitrary plant, that is any $d_P(s)$, $n_Q(s)$ or for a plant subject to parameter uncertainties $(d_P(s) \rightarrow \tilde{d}_P(s),\ n_Q(s) \rightarrow \tilde{n}_Q(s))$ it is necessary that

$$m_r(s) \mid d_C(s) \tag{18a}$$

$$m_d(s) \mid d_C(s). \tag{18b}$$

Linear Servomechanisms (cont.)

Since from (7) we have

$$m(s) \mid d_C(s) \tag{19}$$

and thus the controller must have the form

$$C(s) = \frac{\alpha(s)}{m(s)\,\beta(s)} \tag{20}$$

where the coefficients of $\alpha(s)$ and $\beta(s)$ are to be chosen to stabilize the closed loop, or equivalently, to render $d_{cl}(s)$ in (16) Hurwitz. Writing out

$$d_{cl}(s) = m(s)\,\beta(s)\,d_P(s) + \alpha(s)\,n_P(s) \tag{21}$$

we see that $d_{cl}(s)$ cannot be made Hurwitz by any choice of $\alpha(s)$, $\beta(s)$ if $m(s)$ and $n_P(s)$ have a common RHP root, or for that matter if $n_P(s)$ and $d_P(s)$ have a common RHP root.

Linear Servomechanisms (cont.)

It can be shown that when $n_P(s)$ and $m(s)\, d_P(s)$ are coprime, there always exist $\alpha(s)$, $\beta(s)$ such that $d_{cl}(s)$ is Hurwitz. In general it is also required that $C(s)$ be proper, that is

$$\text{degree}\, \alpha(s) \leq \text{degree}\, \beta(s) + \text{degree}\, m(s) \tag{22}$$

in order to avoid pure differentiation.

A low order, and thus low complexity controller can be designed if $\beta(s)$ can be chosen to be of low degree. The problem of choosing a low order controller is a relatively difficult problem and does not have known general solution, even for simple cases such as a first order or second order controllers. A high order solution exists and corresponds to the case where $\beta(s)$ and $\alpha(s)$ are of high enough degree that all closed loop characteristic can be arbitrarily assigned. This result is stated below as a theorem.

Linear Servomechanisms (cont.)

Theorem

The servomechanism problem where $r(t)$ must be tracked and $d(t)$ rejected with zero steady state error for the class of signals given by (19) can be solved for a plant of order n (degree of $d_P(s) = n$) with a proper controller if and only if $d_P(s)\, m(s)$ and $n_P(s)$ have no common roots. If degree $\beta(s)$ is chosen to be $n-1$ and degree $m(s) = m$ then the $m + 2n - 1$ characteristic roots of the closed loop system may be arbitrarily assigned by choosing the coefficients of $\alpha(s)$ and $\beta(s)$ with

$$\text{degree}\,\alpha(s) \leq \text{degree}\,\beta(s) + m. \tag{23}$$

The next example shows how this theorem may be applied.

Linear Servomechanisms (cont.)

Example

Consider the servomechanism problem with following specifications:

Plant :

$$Y(s) = \frac{1}{s-1}U(s) + \frac{1}{s-1}D(s) \tag{24}$$

References : Steps

Disturbances : Steps and sinusoids of radian frequency $\omega_0 = 1$.

Linear Servomechanisms (cont.)

We see that $m_r(s) = s$, $m_d(s) = s(s^2+1)$. Therefore $m(s) = s(s^2+1)$. Since the plant is of order $n = 1$, we may take $\beta(s) = 1$ (monic polynomial of degree 0) and $\alpha(s) = \alpha_0 + \alpha_1\, s + \alpha_2\, s^2 + \alpha_3\, s^3$. The controller transfer function is

$$C(s) = \frac{\alpha_0 + \alpha_1\, s + \alpha_2\, s^2 + \alpha_3\, s^3}{s(s^2+1)} \tag{25}$$

and the closed loop characteristic polynomial

$$\begin{aligned} d_{cl}(s) &= s(s^2+1)(s-1) + (\alpha_0 + \alpha_1\, s + \alpha_2\, s^2 + \alpha_3\, s^3) \\ &= s^4 + (\alpha_3 - 1)s^3 + (\alpha_2 + 1)s^2 + (\alpha_1 - 1)s + \alpha_0. \end{aligned} \tag{26}$$

Linear Servomechanisms (cont.)

To proceed we need to choose the 4 closed loop characteristic roots to be assigned by the controller (25). For this purposes of this example we choose these as

$$\lambda_1 = -1, \quad \lambda_2 = -1 + j, \quad \lambda_3 = -1 - j, \quad \lambda_4 = -2 \qquad (27)$$

corresponding to the monic polynomial

$$(s+1)((s+1)^2+1)(s+2) = s^4 + 5s^3 + 10s^2 + 10s + 4. \qquad (28)$$

Equating coefficients in (27) and (28) we obtain

$$\alpha_3 = 6, \quad \alpha_2 = 9, \quad \alpha_1 = 11, \quad \alpha_0 = 4 \qquad (29)$$

and the corresponding controller

$$C(s) = \frac{4 + 11s + 9s^2 + 6s^3}{s(s^2+1)}. \qquad (30)$$

Linear Servomechanisms (cont.)

Example

Suppose that the plant is

$$Y(s) = \frac{s - z}{(s-1)^2} U(s) + \frac{1}{(s-1)^2} D(s) \tag{31}$$

and $r(t)$ consists of steps and ramps and $d(t)$ consists of steps and sinusoids of radian frequency $\omega_0 = 1$.

Linear Servomechanisms (cont.)

Then the controller would be of the form

$$\begin{aligned} C(s) &= \frac{\alpha_0 + \alpha_1\, s + \alpha_2\, s^2 + \alpha_3\, s^3 + \alpha_4\, s^4 + \alpha_5\, s^5}{s^2(s^2+1)(s+\beta_0)} \\ &= \frac{\alpha(s)}{s^2(s^2+1)(s+\beta_0)} \end{aligned} \tag{32}$$

and the seven coefficients $(\alpha_0, \ldots, \alpha_5, \beta_0)$ would be determined by chossing 7 closed loop characteristic roots $\lambda_1, \ldots, \lambda_7$, forming the design polynomial

$$a(s) = \Pi_{i=1}^{7}(s - \lambda_i) \tag{33}$$

and equating the coefficients of

$$d_{cl}(s) = (s-1)^2 s^2 (s^2+1)(s+\beta_0) + (s-z)\alpha(s) \tag{34}$$

and those of $a(s)$. Note that a solution can exist if and only if $z \neq 0$ and $z \neq 1$, that is the RHP zeros of the plant do not coincide with plant poles or signal poles.

Exercises

Exercise 1

Consider the feedback system,

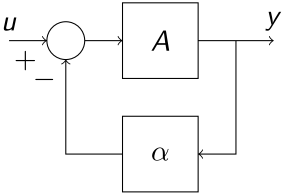

Figure 2:

Let A be a real number gain which is subject to 30% uncertainty. Find the value of α so that the closed loop has 3% uncertainty.

Exercises (cont.)

Exercise 2

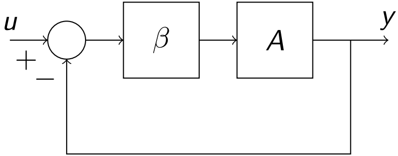

Figure 3:

Repeat the problem in Exercise 1 with the system in Figure 3. Find β to make the system robust.

Exercises (cont.)

Exercise 3

Consider the feedback system,

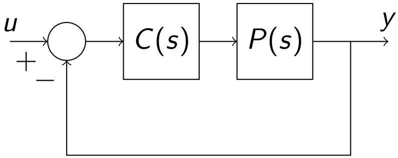

Figure 4:

Let $C(s)$ be a proportional gain controller. That is, we have $C(s) = K$ where K is a real number gain. Find the range of the gain of the controller that stabilize the following plants.

a) $P(s) = \dfrac{9}{s-3}$

Exercises (cont.)

b) $P(s) = \dfrac{s+1}{s-1}$

c) $P(s) = \dfrac{s+1}{(s-1)(s+2)}$

d) $P(s) = \dfrac{s+1}{s(s-1)(s+2)}$

Exercises (cont.)

Exercise 4

For the plant $P(s)$'s in Exercise 3, use the proportional and integral controller. That is, now $C(s) = \dfrac{K_P\, s + K_I}{s}$ with $K_I \neq 0$. First, find the PI gain for each of the systems that stablizes the plant. Second, find the steady state error for

a) unit step input,

b) unit ramp input.

Exercises (cont.)

Exercise 5

(MATLAB) For the plant $P(s)$'s in Exercise 3, design a *PI* controller and plot the step reference tracking and the step disturbance rejection.