ECEN 605

LINEAR SYSTEMS

Lecture 24

Matrix Fraction Description I
– Hermite Forms and GCD's

Right and Left Factorization

In some cases an easier approach to finding a minimal realization is through a factorization of the rational transfer matrix $G(s)$ as a "ratio" of polynomial matrices representing the "numerator" and "denominator."

Any $r \times m$ rational matrix transfer function $G(s)$ can be written as follows:

$$\begin{aligned} \underbrace{G(s)}_{r\times m} &= \underbrace{N_R(s)}_{r\times m}\underbrace{D_R^{-1}(s)}_{m\times m} \qquad \text{(right MFD)} \\ &= \underbrace{D_L^{-1}(s)}_{r\times r}\underbrace{N_L(s)}_{r\times m} \qquad \text{(left MFD)} \end{aligned}$$

where $N_R(s)$, $D_R(s)$, $N_L(s)$, $D_L(s)$ are *polynomial matrices*.

Right and Left Factorization (cont.)

Example

$$
\begin{aligned}
G(s) &= \begin{bmatrix} \frac{n_{11}(s)}{d_{11}(s)} & \cdots & \frac{n_{1m}(s)}{d_{1m}(s)} \\ \vdots & & \vdots \\ \frac{n_{r1}(s)}{d_{r1}(s)} & \cdots & \frac{n_{rm}(s)}{d_{rm}(s)} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{n}_{11}(s) & \cdots & \tilde{n}_{1m}(s) \\ \vdots & & \vdots \\ \tilde{n}_{r1}(s) & \cdots & \tilde{n}_{rm}(s) \end{bmatrix} \begin{bmatrix} d_1(s) & & \\ & \ddots & \\ & & d_m(s) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \bar{d}_1(s) & & \\ & \ddots & \\ & & \bar{d}_r \end{bmatrix}^{-1} \begin{bmatrix} \bar{n}_{11}(s) & \cdots & \bar{n}_{1m}(s) \\ \vdots & & \vdots \\ \bar{n}_{r1}(s) & \cdots & \bar{n}_{rm}(s) \end{bmatrix}.
\end{aligned}
$$

Right and Left Factorization (cont.)

The right decomposition can lead to a controllable realization and the left decomposition to an observable realization of

$$\begin{aligned} \text{order} &= \text{degree } [\det(D_R(s))] \quad \text{or} \\ &= \text{degree } [\det(D_L(s))]\,, \end{aligned}$$

respectively as we show in the example below.

Right and Left Factorization (cont.)

Example

$$G(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s+1} & \frac{1}{s} \\ \frac{1}{s+1} & \frac{1}{s} & \frac{1}{s-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1+s}{s^2-1} & \frac{s}{s^2+s} & \frac{-1+s}{s^2-s} \\ \frac{-1+s}{s^2-1} & \frac{1+s}{s^2+s} & \frac{s}{s^2-s} \end{bmatrix}}_{A(s)}$$

$$= \underbrace{\begin{bmatrix} \frac{s+s^2}{s^3-s} & \frac{-s+s^2}{s^3-s} & \frac{s^2-1}{s^3-s} \\ \frac{-s+s^2}{s^3-s} & \frac{-1+s^2}{s^3-s} & \frac{s+s^2}{s^3-s} \end{bmatrix}}_{B(s)}.$$

Right and Left Factorization (cont.)

Let us observe $A(s)$. It can be decomposed as follows:

$$A(s) = \begin{bmatrix} 1+s & s & -1+s \\ -1+s & 1+s & s \end{bmatrix} \begin{bmatrix} s^2-1 & & \\ & s^2+s & \\ & & s^2-s \end{bmatrix}^{-1}.$$

Right and Left Factorization (cont.)

A state space realization can be easily obtained form the above as follows:

$$
A_c = \left[\begin{array}{cc:cc:cc}
0 & 1 & & & & \\
1 & 0 & & & & \\
\hdashline
 & & 0 & 1 & & \\
 & & 0 & -1 & & \\
\hdashline
 & & & & 0 & 1 \\
 & & & & 0 & 1
\end{array}\right]
\qquad
B_c = \left[\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
\hdashline
0 & 0 & 0 \\
0 & 1 & 0 \\
\hdashline
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

$$
C_c = \left[\begin{array}{cc:cc:cc}
1 & 1 & 0 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1 & 0 & 1
\end{array}\right].
$$

As seen, the above realization is controllable of order 6 which is equal to the degree of the determinant of $D_R(s)$.

Right and Left Factorization (cont.)

Now let us observe the MFD $B(s)$. It can be decomposed as:

$$B(s) = \begin{bmatrix} s^3 - s & 0 \\ 0 & s^3 - s \end{bmatrix}^{-1} \begin{bmatrix} s + s^2 & -s + s^2 & -1 + s^2 \\ -s + s^2 & -1 + s^2 & s + s^2 \end{bmatrix}.$$

A state space realization may be

$$A_o = \left[\begin{array}{ccc:ccc} 0 & 0 & 0 & & & \\ 1 & 0 & 1 & & & \\ 0 & 1 & 0 & & & \\ \hdashline & & & 0 & 0 & 0 \\ & & & 1 & 0 & 1 \\ & & & 0 & 1 & 0 \end{array}\right] \qquad B_o = \left[\begin{array}{c:c:c} 0 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \\ \hdashline 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{array}\right]$$

$$C_o = \left[\begin{array}{ccc:ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Right and Left Factorization (cont.)

This realization is observable and of order is equal to the degree of the determinant of $D_L(s)$.
To get *minimality* we must extend the concept of *coprimeness* of scalar polynomials to polynomial matrices.

Hermite Forms and GCD's

We first need the concept of Hermite form of a polynomial matrix.

$$\text{Row Hermite form} = \text{Upper Triangular form}$$

$$\begin{bmatrix} a_{11}(s) & a_{12}(s) & \cdots & \cdots & \cdots \\ 0 & a_{22}(s) & \vdots & & \vdots \\ \vdots & \ddots & & & \vdots \\ 0 & & 0 & a_{rr}(s) & a_{r,r+1}(s) \\ \vdots & & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

with $a_{ii}(s)$ of degree higher than all others in i^{th} column.

Hermite Forms and GCD's (cont.)

The construction can be done by row elementary operations corresponding to multiplication with unimodular matrices on the left constructed from quotients and remainders using the Euclidean division algorithm.

Hermite Forms and GCD's (cont.)

Example (Reduction to Hermite Form)

$$\begin{bmatrix} s^2 & 0 \\ 0 & s^2 \\ 1 & s+1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & s+1 \\ 0 & s^2 \\ s^2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & s+1 \\ 0 & s^2 \\ 0 & -s^2(s+1) \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & s+1 \\ 0 & s^2 \\ 0 & 0 \end{bmatrix}$$

The corresponding unimodular matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s+1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s^2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & s+1 & s^2 \end{bmatrix}.$$

Construction of GCRD's

To construct a GCRD of $N(s), D(s)$ we find a unimodular $U(s)$ to carry out the row compression

$$\underbrace{\begin{bmatrix} \underbrace{U_{11}(s)}_{m\times m} & U_{12}(s) \\ U_{21}(s) & \underbrace{U_{22}(s)}_{p\times p} \end{bmatrix}}_{U(s)} \begin{bmatrix} \underbrace{D(s)}_{m\times m} \\ \underbrace{N(s)}_{p\times m} \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}$$

Then, $R(s)$ is a GCRD.

Construction of GCRD's (cont.)

Proof

Let

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix}^{-1} = \begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix}$$

Then,

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} V_{11}(s)R(s) \\ V_{21}(s)R(s) \end{bmatrix}$$

and it is easy to see that $R(s)$ is a common right divisor.

Construction of GCRD's (cont.)

Consequently, we have the following matrix equation.

$$R(s) = U_{11}(s)D(s) + U_{12}(s)N(s). \tag{1}$$

If $R_1(s)$ is another common right divisor, then

$$D(s) = D_1(s)R_1(s), \qquad N(s) = N_1(s)R_1(s)$$

and we have

$$R(s) = [U_{11}(s)D_1(s) + U_{12}(s)N_1(s)]\, R_1(s)$$

so that $R_1(s)$ is a right divisor of $R(s)$ and so $R(s)$ is a GCRD.

Construction of GCRD's (cont.)

If $R_1(s)$, $R_2(s)$ are two GCRDs, then

$$R_1(s) = W_2(s)R_2(s), \qquad R_2(s) = W_1(s)R_1(s)$$

$$R_1(s) = W_2(s)W_1(s)R_1(s).$$

If $R_1(s)$ is nonsingular, then $R_2(s)$ is nonsingular and $R_1(s)$, $R_2(s)$ can only differ by a unimodular factor. If $R_1(s)$ is unimodular then $R_2(s)$ is also. If $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ has full column rank, all GCRD's are nonsingular. □

Construction of GCRD's (cont.)

Remark (Nonuniqueness of GCRDs: Summary)

Notice that by carrying out the elementary operations to reduce to Hermite form in difference orders we may get different matrices $U(s)$ and hence different GCRDs. However, any two GCRDs, $R_1(s)$ and $R_2(s)$, say, must be related (by definition) as

$$R_1(s) = W_2(s)R_2(s), \qquad R_2(s) = W_1(s)R_1(s), \qquad W_i(s) \text{ polynomial.}$$

Construction of GCRD's (cont.)

Since

$$R_1(s) = W_2(s)W_1(s)R_1(s)$$

it follows that

1) If $R_1(s)$ is nonsingular, then the $W_i(s)$, $i = 1, 2$, must be unimodular, and hence the GCRD $R_2(s)$ is also nonsingular. That is, *if one GCRD is nonsingular, then all GCRDs must be so, and they can only differ by a unimodular (left) factor.*

2) *If a gcrd is unimodular, then all gcrds must be unimodular.*

Construction of GCRD's (cont.)

Remark (Nonsingular GCRDs)

If

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \quad \text{has full column rank,}$$

then *all* GCRDs of $(N(s), D(s))$ must be nonsingular and can differ only by unimodular (left) factors.

Construction of GCRD's (cont.)

Definition

$(N_R(s), D_R(s))$ is *right coprime* if and only if all its GCRDs are unimodular.

Similarly,

$(N_L(s), D_L(s))$ is *left coprime* if and only if all its GCLDs are unimodular.