ECEN 605

LINEAR SYSTEMS

Lecture 25

Matrix Fraction Description II
– Bezout Identity

Bezout Identity

Theorem

$(N(s), D(s))$ is right coprime if and only if there exist $X(s)$ and $Y(s)$ such that

$$X(s)N(s) + Y(s)D(s) = I$$

Bezout Identity (cont.)

Proof

From

$$R(s) = U_{11}(s)D(s) + U_{12}(s)N(s), \tag{1}$$

if $R(s)$ is a GCRD, we have

$$R(s) = \hat{X}(s)N(s) + \hat{Y}(s)D(s).$$

Furthermore, if $N(s)$ and $D(s)$ are coprime, then $R(s)$ is unimodular. Thus,

$$\begin{aligned} I &= \underbrace{R^{-1}(s)\hat{X}(s)}_{X(s)} N(s) + \underbrace{R^{-1}(s)\hat{Y}(s)}_{Y(s)} D(s); \quad R^{-1}(s) \text{ is polynomial} \\ &= X(s)N(s) + Y(s)D(s). \end{aligned}$$

Bezout Identity (cont.)

Conversely, if there exist $X(s)$ and $Y(s)$ satisfying

$$I = X(s)N(s) + Y(s)D(s),$$

let $R(s)$ be any gcrd. Then

$$N(s) = \bar{N}(s)R(s), \qquad D(s) = \bar{D}(s)R(s)$$

and

$$\begin{aligned} I &= X(s)\bar{N}(s)R(s) + Y(s)\bar{D}(s)R(s) \\ &= \left(X(s)\bar{N}(s) + Y(s)\bar{D}(s)\right) R(s). \end{aligned}$$

Bezout Identity (cont.)

It follows that

$$R^{-1}(s) = X(s)\bar{N}(s) + Y(s)\bar{D}(s) \qquad \text{(polynomial)}$$

Since $R^{-1}(s)$ is unimodular, $(N(s), D(s))$ is coprime. $\square$

Bezout Identity (cont.)

Lemma

$(N(s), D(s))$ is coprime iff

$$\operatorname{rank} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = m, \qquad \text{for all } s.$$

Bezout Identity (cont.)

Lemma

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}$$

If $D(s)$ is nonsingular, then

(a) $U_{22}(s)$ is nonsingular,

(b) $N(s)D^{-1}(s) = -U_{22}^{-1}(s)U_{21}(s)$,

(c) $(U_{21}(s), U_{22}(s))$ is left coprime, and

(d) $\deg(\det[D(s)]) = \deg(\det[U_{22}(s)])$ if $(N(s), D(s))$ is coprime.

Bezout Identity (cont.)

Proof

(a) If $U_{22}(s)$ is singular, then there exists nonzero $\alpha(s)$ such that

$$\alpha(s)U_{22}(s) = 0.$$

It follows that

$$\alpha(s)U_{21}(s)D(s) + \alpha(s)U_{22}(s)N(s) = 0$$

which implies that

$$\alpha(s)\begin{bmatrix} U_{21}(s) & U_{22}(s) \end{bmatrix} = [0 \quad 0].$$

This means that $U(s)$ is not unimodular. Therefore, $U_{22}(s)$ is nonsingular.

Bezout Identity (cont.)

(b)

$$U_{21}(s)D(s) + U_{22}(s)N(s) = 0$$

directly implies that

$$\begin{array}{rcl} U_{21}(s)D(s) & = & -U_{22}(s)N(s) \\ & \Downarrow & \\ U_{22}^{-1}(s)U_{21}(s) & = & -N(s)D^{-1}(s). \end{array}$$

Bezout Identity (cont.)

(c) From Lemma 2, since

$$\begin{bmatrix} U_{21}(s) & U_{22}(s) \end{bmatrix} \quad \text{has full row rank for all } s,$$

$(U_{21}(s), U_{22}(s))$ is left coprime.

Bezout Identity (cont.)

(d) Notice the following two matrix identities[1]:

$$\det \begin{bmatrix} A & D \\ C & B \end{bmatrix} = \det [A] \det \left[B - CA^{-1}D \right] \tag{2}$$

and

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix} \tag{3}$$

where

$$\begin{aligned} \Delta &= B - CA^{-1}D \\ E &= A^{-1}D \\ F &= CA^{-1}. \end{aligned}$$

Recall that

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}$$

Bezout Identity (cont.)

and

$$\begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix} \begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

then we have

$$D(s) = V_{11}(s)R(s).$$

Thus, since $\deg(\det[R(s)]) = 0$,

$$\begin{aligned} \deg(\det[D(s)]) &= \deg(\det[V_{11}(s)]) + \deg(\det[R(s)]) \\ &= \deg(\det[V_{11}(s)]) \end{aligned} \quad (4)$$

Using the formulae given in eq. (2), we have

$$\begin{aligned} \det[V(s)] &= \det[V_{11}(s)] \det\left[V_{22}(s) - V_{21}(s)V_{11}^{-1}(s)V_{12}(s)\right] \\ &= \det[V_{11}(s)] \det\left[U_{22}^{-1}\right] \quad (\textit{see eq. } (3)) \\ &= \det[V_{11}(s)] \frac{1}{\det[U_{22}(s)]} \end{aligned}$$

Bezout Identity (cont.)

Thus,

$$\det\left[U(s)\right] = \frac{\det\left[U_{22}(s)\right]}{\det\left[V_{11}(s)\right]}.$$

Here, we know that $\det\left[U(s)\right] = \mathrm{constant}$ (i.e., unimodular). This leads

$$\begin{aligned}
\det\left[V_{11}(s)\right] &= \frac{\det\left[U_{22}(s)\right]}{\mathrm{constant}} \\
&\Downarrow \\
\deg\left(\det\left[V_{11}(s)\right]\right) &= \deg\left(\det\left[U_{22}\right]\right) \\
&= \deg\left(\det\left[D(s)\right]\right). \quad (\textit{see eq.}\ (4))
\end{aligned}$$

$\square$

[1] T. Kailath, *Linear Systems*, Prentice-Hall, 1980, p. 680, p. 656

Generalized Bezout Identity

Theorem

$(N_R(s), D_R(s))$ is right coprime with $\det[D_R(s)] \neq 0$*, then there exist*
$(X(s), Y(s), X^*(s), Y^*(s))$ and $(N_L(s), D_L(s))$ such that

$$\begin{bmatrix} -X(s) & Y(s) \\ D_L(s) & N_L(s) \end{bmatrix} \begin{bmatrix} -N_R(s) & X^*(s) \\ D_R(s) & Y^*(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{5}$$

and

$$\begin{bmatrix} -N_R(s) & X^*(s) \\ D_R(s) & Y^*(s) \end{bmatrix} \begin{bmatrix} -X(s) & Y(s) \\ D_L(s) & N_L(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{6}$$

Moreover, the block matrices will be unimodular.

Generalized Bezout Identity (cont.)

Proof

From the 1st Theorem, we know that $(N_R(s), D_R(s))$ being right coprime implies that there exist polynomial matrices $(X(s), Y(s))$ such that

$$X(s)N_R(s) + Y(s)D_R(s) = I. \tag{7}$$

From the 3rd Lemma, we also know that there will exist left coprime polynomial matrices $(N_L(s), D_L(s))$ such that

$$D_L^{-1}(s)N_L(s) = N_R(s)D_R^{-1}(s).$$

Generalized Bezout Identity (cont.)

Furthermore, the left coprimeness of $(N_L(s), D_L(s))$ implies the existence of polynomial matrices $(\bar{X}(s), \bar{Y}(s))$ such that

$$D_L(s)\bar{X}(s) + N_L(s)\bar{Y}(s) = I. \tag{8}$$

Putting eqs. (7) and (8) together, we have

$$\begin{bmatrix} -X(s) & Y(s) \\ D_L(s) & N_L(s) \end{bmatrix} \begin{bmatrix} -N_R(s) & \bar{X}(s) \\ D_R(s) & \bar{Y}(s) \end{bmatrix} = \begin{bmatrix} I & Q(s) \\ 0 & I \end{bmatrix}$$

where

$$Q(s) = -X(s)\bar{X}(s) + Y(s)\bar{Y}(s).$$

Generalized Bezout Identity (cont.)

Thus,

$$\begin{bmatrix} -X(s) & Y(s) \\ D_L(s) & N_L(s) \end{bmatrix} \begin{bmatrix} -N_R(s) & \bar{X}(s) \\ D_R(s) & \bar{Y}(s) \end{bmatrix} \begin{bmatrix} I & -Q(s) \\ 0 & I \end{bmatrix} =$$

$$\begin{bmatrix} -X(s) & Y(s) \\ D_L(s) & N_L(s) \end{bmatrix} \begin{bmatrix} -N_R(s) & N_R(s)Q(s) + \bar{X}(s) \\ D_R(s) & -D_R(s)Q(s) + \bar{Y}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Generalized Bezout Identity (cont.)

If we set

$$
\begin{aligned}
X^*(s) &= N_R(s)Q(s) + \bar{X}(s) \\
Y^*(s) &= -D_R(s)Q(s) + \bar{Y}(s),
\end{aligned}
$$

we have eq. (5). Eq. (6) follows by using the fact that $CD = I$ implies that $DC = I$ when (C,D) are square constant or polynomial matrices. Finally, we need to show the block matrices will be unimodular. This part is obvious from eqs. (5) and (6) that the inverse of each block matrix is a polynomial matrix.

Generalized Bezout Identity (cont.)

Remark

To determine properness, the same definitions in the scalar case hold here.

Proper:	$\lim_{s\to\infty} H(s) < \infty$
Strictly Proper:	$\lim_{s\to\infty} H(s) = 0$

Generalized Bezout Identity (cont.)

Definition

The highest degree of all the entries of the vector is called the *degree of a polynomial vector* (or *column degree*).

Lemma

If $H(s)$ strictly proper (proper), and

$$H(s) = N(s)D^{-1}(s)$$

then every column of $N(s)$ has (column) degree strictly less than (less than or equal to) that of the corresponding column of $D(s)$.

Generalized Bezout Identity (cont.)

Proof

Since $N(s) = H(s)D(s)$, we have

$$\begin{bmatrix} h_{11}(s) & \cdots & h_{1m}(s) \\ \vdots & & \vdots \\ h_{r1}(s) & \cdots & h_{rm}(s) \end{bmatrix} \begin{bmatrix} \cdots & d_{j1}(s) & \cdots \\ & \vdots & \\ \cdots & d_{jm}(s) & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & n_{j1}(s) & \cdots \\ & \vdots & \\ \cdots & n_{jr}(s) & \cdots \end{bmatrix}.$$

Clearly, $H(s)$ being strictly proper (proper) implies that the column degree of

$$\begin{bmatrix} n_{j1}(s) & n_{j2}(s) & \cdots & n_{jr}(s) \end{bmatrix}^T$$

is strictly less that (less than or equal to) the column degree of

$$\begin{bmatrix} d_{j1}(s) & d_{j2}(s) & \cdots & d_{jm}(s) \end{bmatrix}^T.$$

□

Generalized Bezout Identity (cont.)

Example (Converse is not true.)

Let

$$N(s) = \begin{bmatrix} 2s^2 + 1 & 2 \end{bmatrix}, \qquad D(s) = \begin{bmatrix} s^3 + s & s \\ s^2 + s + 1 & 1 \end{bmatrix}$$

The column degrees of $N(s)$ are 2 and 0, and the column degrees of corresponding vectors in $D(s)$ are 3 and 1. Each of the column degree of $N(s)$ is strictly less than the column degree of the corresponding column of $D(s)$.

Generalized Bezout Identity (cont.)

So, is $H(s)$ strictly proper?

$$H(s) = N(s)D^{-1}(s) = \begin{bmatrix} \dfrac{-s^2+s}{s^2+s-1} & \dfrac{s^3+s-1}{s^2+s-1} \end{bmatrix}$$

It shows that $H(s)$ is improper.

To make the converse true, we need the concept of column reduced polynomial matrices.

Column Reduced Matrices

Let k_i be the degree of the i^{th} column of $D(s)$. Then

$$\deg\left(\det\left[D(s)\right]\right) \leq \sum_{i=1}^{m} k_i.$$

We say that $D(s)$ is *column reduced* if the equality holds.

Column Reduced Matrices (cont.)

Example

$$D(s) = \begin{bmatrix} s^3 + s & s + 2 \\ s^2 + s + 1 & 1 \end{bmatrix}$$

It is easy to see that $k_1 = 3$ and $k_2 = 1$, and the sum of the column degree is 4. However,

$$\deg \left(\det \begin{bmatrix} s^3 + s & s + 2 \\ s^2 + s + 1 & 1 \end{bmatrix} \right) \neq 4.$$

Thus, $D(s)$ is not column reduced. This is due to the coefficient matrix of the highest terms in each column is singular.

$$\begin{aligned} D(s) &= \begin{bmatrix} s^3 + s & s + 2 \\ s^2 + s + 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s^3 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} s & 2 \\ s^2 + s + 1 & 1 \end{bmatrix} \end{aligned}$$

Column Reduced Matrices (cont.)

As seen above, we can always write

$$D(s) = D_{hc}S(s) + L(s)$$

where

$$S(s) = \begin{bmatrix} s^{k_1} & & & \\ & s^{k_2} & & \\ & & \ddots & \\ & & & s^{k_m} \end{bmatrix}$$

and D_{hc} is the highest column degree coefficient matrix. Thus,

$$\det[D(s)] = \det[D_{hc}]\, s^{\sum_i k_i} + \text{terms of lower degree in } s.$$

Column Reduced Matrices (cont.)

Lemma

If $D(s)$ is column reduced, $H(s) = N(s)D^{-1}(s)$ is strictly proper (proper) iff each column degree of $N(s)$ is less than (less than or equal to) the column degree of the corresponding column of $D(s)$.

Column Reduced Matrices (cont.)

Proof

We only prove for the case of strictly proper and the case of a proper matrix is an obvious generalization. We first need to show that if $D(s)$ is column reduced and each column degree of $N(s)$ is less than the corresponding column degree of $D(s)$, then $H(s)$ is strictly proper.

Column Reduced Matrices (cont.)

Using Cramer's rule, we know that ij^{th} entry of $H(s)$ can be written as

$$h_{ij}(s) = \frac{\det\left[D^{ij}(s)\right]}{\det\left[D(s)\right]} \tag{9}$$

where

$$D^{ij}(s) = \begin{bmatrix} d_{11}(s) & \cdots & \cdots & \cdots & d_{1m}(s) \\ \vdots & & & & \vdots \\ d_{j-1,1}(s) & \cdots & \cdots & \cdots & d_{j-1,m}(s) \\ n_{i1}(s) & \cdots & \cdots & \cdots & n_{im}(s) \\ d_{j+1,1}(s) & \cdots & \cdots & \cdots & d_{j+1,m}(s) \\ \vdots & & & & \vdots \\ d_{m1}(s) & \cdots & \cdots & \cdots & d_{mm}(s) \end{bmatrix}.$$

Column Reduced Matrices (cont.)

Now write

$$D^{ij}(s) = D^{ij}_{hc} S(s) + L^{ij}(s).$$

Note that D^{ij}_{hc} is identical to D_{hc} except that the j^{th} row is now replaced by zero since each entry of the j^{th} row is of lower degree than the corresponding entry of the j^{th} row of $D(s)$. Hence, D^{ij}_{hc} is singular, while D_{hc} is nonsingular. It implies that

$$\deg\left(\det\left[D^{ij}(s)\right]\right) < \sum_{1}^{m} k_i = \deg\left(\det\left[D(s)\right]\right) = \deg\left(\det\left[S(s)\right]\right).$$

Therefore, $h^{ij}(s)$ is strictly proper (see eq. (9)) and hence so is the matrix $H(s)$. Next, we need to prove that if $H(s)$ is strictly proper and $D(s)$ is column reduced, then each column degree of $N(s)$ is less than the corresponding column degree of $D(s)$. This follows directly from Lemma 7. $\square$

Column Reduced Matrices (cont.)

Remark

$D(s)$ is column reduced iff $\det[D_{hc}] \neq 0$.

Lemma

If $D(s)$ is not column reduced there exists unimodular $U(s)$ so that $D(s)U(s)$ is column reduced.

Column Reduced Matrices (cont.)

Example

$$D(s) = \begin{bmatrix} (s+1)^2(s+2)^2 & -(s+1)^2(s+2) \\ 0 & s+2 \end{bmatrix}$$

$k_1 = 4$, $k_2 = 3$, and

$$D(s) = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}_{D_{hc}} \begin{bmatrix} s^4 & 0 \\ 0 & s^3 \end{bmatrix} + L(s)$$

$D(s)$ is not column reduced since D_{hc} is singular.

Column Reduced Matrices (cont.)

So, there must exist $U(s)$ to make it column reduced. Let us first take

$$U_1(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix},$$

then we have

$$D_1(s) = D(s)U_1(s) = \begin{bmatrix} 2s^3 + 8s^2 + 10s + 4 & -s^3 + 4s^2 + 5s + 2 \\ s^2 + 2s & s + 2 \end{bmatrix}.$$

Column Reduced Matrices (cont.)

Now let us observe the highest degree coefficient matrix of $D_1(s)$.

$$D_{1,hc} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

This shows that it is still not column reduced. So we take

$$U_2(s) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

and we have

$$D_2(s) = D_1(s)U_2(s) = \begin{bmatrix} 0 & s^3 + 4s^2 + 5s + 2 \\ s^2 + 4s + 4 & s + 2 \end{bmatrix}.$$

Since

$$D_{2,hc} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is nonsingular, $D_2(s)$ is column reduced.