ECEN 605

LINEAR SYSTEMS

Lecture 26

Matrix Fraction Description III
– Minimal Realization from MFD's

Minimal Realization from MFD's

With the results of the last lecture we have

Lemma

Let

$$G(s) = N(s)D^{-1}(s)$$

with $D(s)$ column reduced, and let

$$\deg [n_i(s)] < \deg [d_i(s)]$$

where $n_i(s)$ and $d_i(s)$ are the i^{th} columns of $N(s)$ and $D(s)$, respectively. Then $G(s)$ is strictly proper.

Minimal Realization from MFD's (cont.)

Using this result, we can give the following realization. Let $G(s) = N(s)D^{-1}(s)$ be strictly proper, and write

$$D(s) = D_{hc}S(s) + D_{lc}\psi(s)$$

where

$$S(s) = \begin{bmatrix} s^{k_1} & & & \\ & s^{k_2} & & \\ & & \ddots & \\ & & & s^{k_r} \end{bmatrix}, \qquad \psi(s) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ s & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ s^{k_1-1} & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & s & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & s^{k_2-1} & & 0 \\ & & \ddots & \\ & & & 1 \\ & & & s \\ & & & \vdots \\ 0 & 0 & \cdots & s^{k_r-1} \end{bmatrix}$$

with D_{hc} invertible.

Minimal Realization from MFD's (cont.)

Let

$$B_c^o = \begin{bmatrix} 0 & & & \\ \vdots & & & \\ 0 & & & \\ 1 & & & \\ & 0 & & \\ & \vdots & & \\ & 0 & & \\ & 1 & & \\ & & \ddots & \\ & & & 0 \\ & & & \vdots \\ & & & 0 \\ & & & 1 \end{bmatrix},$$

Minimal Realization from MFD's (cont.)

$$A_c^o = \begin{bmatrix} \begin{matrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{matrix} \end{bmatrix}$$

Minimal Realization from MFD's (cont.)

Remark

Write

$$N(s) = N_{lc}\psi(s).$$

This is always possible because $\deg[n_i(s)]$ *is less than* $\deg[d_i(s)]$*.*

Claim:

$$\begin{aligned} A_c &= A_c^o - B_c^o D_{hl}^{-1} D_{lc} \\ B_c &= B_c^o D_{hl}^{-1} \\ C_c &= N_{lc} \end{aligned}$$

is a controllable realization. If $(N(s), D(s))$ are right coprime, it is also an observable and therefore minimal, realization.

Minimal Realization from MFD's (cont.)

Example

$$G(s) = \underbrace{\begin{bmatrix} s & 0 \\ -s & s^2 \end{bmatrix}}_{N(s)} \underbrace{\begin{bmatrix} 0 & -(s^3+4s^2+5s+2) \\ (s+2)^2 & s+2 \end{bmatrix}}_{D(s)}^{-1}$$

As seen, $k_1 = 2$ and $k_2 = 3$. Thus,

$$D_{hc} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This implies that $D(s)$ is column reduced. Therefore,

$$\deg[n_1(s)] = 1 < \deg[d_1(s)] = 2, \quad \deg[n_2(s)] = 2 < \deg[d_2(s)] = 3.$$

Minimal Realization from MFD's (cont.)

Now we write

$$D(s) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{D_{hc}} \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}}_{S(s)} + \underbrace{\begin{bmatrix} 0 & -4s^2 - 5s - 2 \\ 4s + 4 & s + 2 \end{bmatrix}}_{L(s)}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & -2 & -5 & -4 \\ 4 & 4 & 2 & 1 & 0 \end{bmatrix}}_{D_{lc}} \underbrace{\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \end{bmatrix}}_{\psi(s)}$$

$$N(s) = \begin{bmatrix} s & 0 \\ -s & s^2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}}_{N_{lc}} \underbrace{\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \end{bmatrix}}_{\psi(s)}.$$

Minimal Realization from MFD's (cont.)

Finally,

$$
\begin{aligned}
A_c &= A_c^o - B_c^o D_{hc}^{-1} D_{lc} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & -4 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -5 & -4 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \\
C_c &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

This realization is guaranteed to be controllable. It will be observable if $(N(s), D(s))$ are right coprime.