Transmission Lines

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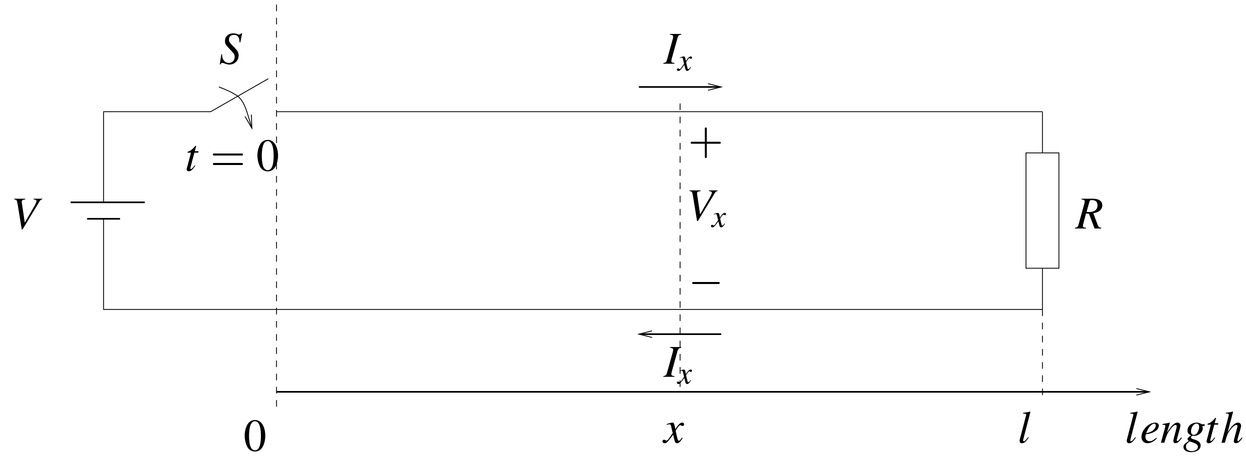
When to Invoke a Transmission Line Model

- Suppose we have a wire. When is it to be viewed as a transmission line?
- **Short answer:** When the *time-of-flight* through the wire is comparable or larger than the rise/fall time of the source signal
- Time of flight is known if we know the velocity of wave propagation and the length of the wire. If the velocity of wave propagation is u and the length of the wire is l, then
  $u = \frac{l}{t}$
- What if we need to model a wire as a transmission line?
  - We simply model the R/L/C of the wire in a **distributed** manner instead of a lumped manner.
  - This simply means using several π or T sections as appropriate (depending on the desired accuracy).
- OK, what about the long answer?

Equations of Propagation

- Consider a wire of length l, connected to a voltage source V, and a switch S as shown below. The load resistance R is connected at the end of the wire.

- At time $t=0$, the switch S is closed.



- The effect of closing S is not felt everywhere immediately

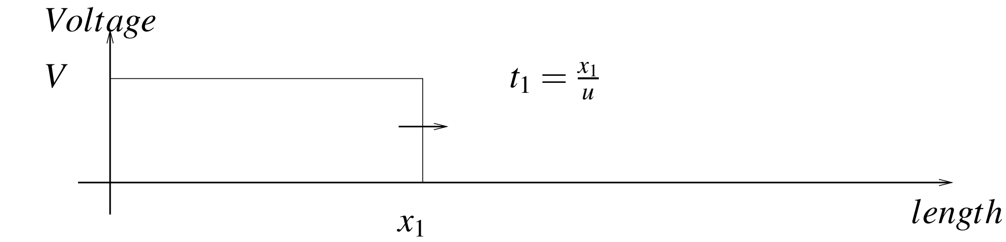
- Rather, this effect propagates from the source to load with a **finite** velocity

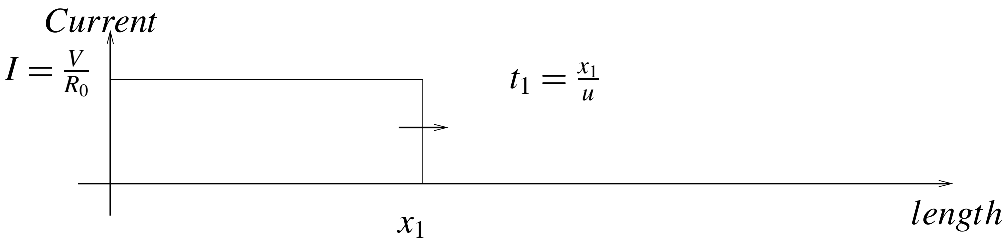
- Assuming uniform cross section, this velocity is given by $u=\frac{1}{\sqrt{LC}}$. Here $u=\frac{c}{\sqrt{\kappa}}$, where κ is the dielectric constant of the medium in which the wire resides. L

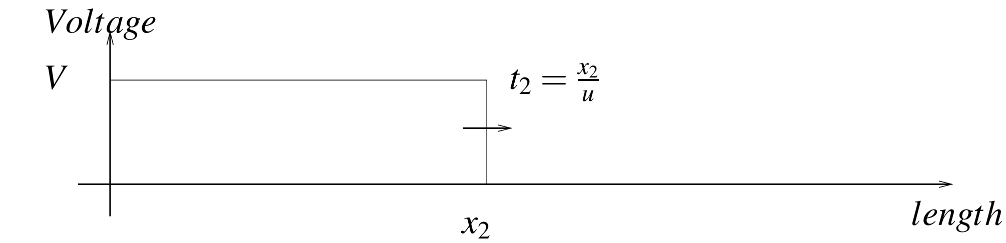
and C are the inductance and capacitance per unit length of wire.

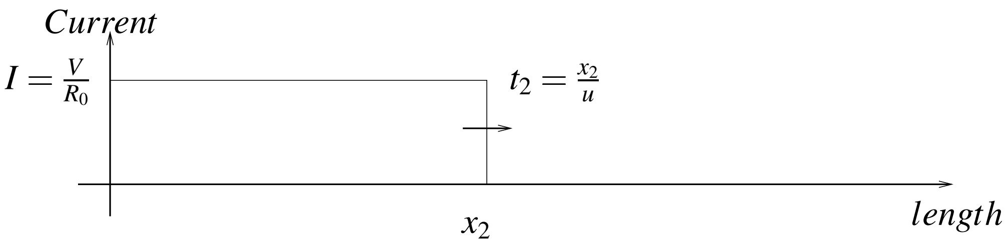
- Note that if L increases, C decreases and vice versa.

- The voltage/current distributions on the wire at times $t_1 = x_1/u$ and $t_2 = x_2/u$ are shown below.









- We introduced a quantity R_0 above. How is it computed and what is it? Lets explain this analytically

- At t_1, the voltage is V for $0 \le x \le x_1$ and 0 for $x_1 < x \le l$

- In other words, a **front** of voltage travels to the right with velocity u.

- Lets say this front moves a distance dx. Then the capacitance to be charged is Cdx (since C is the capacitance per unit length)

- The charge required is $dQ = VCdx$. Hence the current is given by

- $I = \frac{dQ}{dt} = VC\frac{dx}{dt} = VCu = VC\frac{1}{\sqrt{LC}} = V\sqrt{\frac{C}{L}} = \frac{V}{R_0}$

- $R_0 = \frac{L}{C}$ the **Characteristic Impedance** of the line.

- It is a function of the medium in which the wire resides. For PCBs, R_0 is in the 50-75 Ω range.

- The expression for I above is the current for $x = 0$ to wherever the front is. To the right of the front, $I = 0$.

- We say that I_x is positive when current flows to the right in the top wire, and to the left in the bottom wire.

- For all x from 0 to the location of the front (call it x), we have $V_x/I_x = R_0$.

- If the source and load were interchanged, we would have $V_x/I_x = -R_0$ based on our sign convention

Reflections

- If the length l was infinite, then the front would keep moving to the right.

- If l is finite, then but if the load resistor $R = R_0$, then when the front arrives at the load (at time l/u), currents at all points in the wire and the load are V/R_0, and so nothing further happens.

- It is as though the length l was infinite.

- Now lets suppose $R \neq R_0$. When the front arrives at the load (at time l/u), the current in the wire is $V_x/R_x = R_0$. But **at the load,** the relationship $V_l/R_l = R$ **is required.**

- This causes a discontinuity. Thus a **reflection** develops at the load, which moves to the left.

- The amplitude and polarity of this reflected wave are such that the **total** voltage V_l (i.e. sum of the incident and reflected voltage) is related to I_l by the expression $V_l/I_l = R$

- If the incident voltage was V, the reflected voltage will be ρV where ρ is the **reflection coefficient**

- If the incident current is V/R_0, reflected current is $-\rho V/R_0$ based on our sign convention.

- The ratio of the total voltage to total current is R, therefore
$R = \frac{V+\rho V}{V/R_0 - \rho V/R_0}$.

- Solving, we get $\rho = \frac{R/R_0 - 1}{R/R_0 + 1}$. This reflection coefficient ranges from -1 to $+1$.
  - When $R = R_0$, then $\rho = 0$. So no reflections are observed (we agreed that should be the case above).
  - When $R = \infty$ (i.e. the wire is **unterminated**), $\rho = 1$
  - When $R = 0$ (i.e. the wire is short-circuited), $\rho = -1$

- There can be several reflections in this manner. We can compute the waveform at l by accounting for such reflections.

Reflections - An Example

- Consider a wire of length l, with $R_0 = 100\Omega$, $R = 900\Omega$ and a source resistance $R_s = 20\Omega$. Also assume the source voltage is $V = 12$V. We want to determine the waveform at the end of the wire

- First we compute the reflection coefficients.
  - $\rho_R = \frac{\frac{900}{100}-1}{\frac{900}{100}+1} = 0.8$
  - $\rho_S = \frac{\frac{20}{100}-1}{\frac{20}{100}+1} = -0.75$

- A 12 V step appears on the line as a step of voltage $V_1 = 12\frac{R_0}{R_0+R_S} = 10$V

- At time $t = t_d = \frac{l}{u}$, the wave V_1 arrives at the end of the line, where it is reflected as:
  $V_2 = \rho_R \cdot V_1 = 0.8 \cdot 10\text{V} = 8$V

- Therefore the receiver voltage at this time is $V_1 + V_2 = 18$V

- At time $2 \cdot t_d$, the wave V_2 arrives at the source. It is now reflected as:
  $V_3 = \rho_S \cdot V_2 = -0.75 \cdot 8\text{V} = -6$V

- At time $3 \cdot t_d$, the wave V_3 arrives at the end of the line. It is now reflected as:
  $V_4 = \rho_R \cdot V_3 = 0.8 \cdot (-6)\text{V} = -4.8\text{V}$.

- So the receiver voltage at this time is $V_1 + V_2 + V_3 + V_4 = 7.2\text{V}$

- And so on...

- Asymptotically, the voltage at the end of the wire is $V \cdot \frac{R}{R+R_s} = 12 \cdot \frac{900}{920} = 11.739\text{V}$ The waveform at the end of the wire therefore looks like: