ECEN 447 Digital Image Processing

Lecture 7: Mathematical Morphology

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• **Mathematical Morphology** (MM) is a discipline whose roots can be traced to the Ecole de Mines de Paris in the 1960's. It was basically created by the mathematician Georges Matheron and further developed and popularized by his colleague Jean Serra and others.

• Unlike image processing based on linear shift-invariant operators, MM is based on **nonlinear operators**.

• In Biology, morphology means the study of **shape**. MM may be said to be the study of shapes from a mathematical perspective. For that reason, it depends heavily on concepts from **set theory**.

• An attractive feature of MM is that all its numerous operators and algorithms are based on the composition of a small number of building blocks, called **dilation**, **erosion**, **opening**, and **closing**.
Brief Review of Set Theory

- **Set inclusion**: given two sets $A$ and $B$, we have $A \subseteq B$ if
  \[ x \in A \implies x \in B \]

- **Set intersection**: given a family of sets $A_i, i \in I$
  \[ x \in \bigcap_{i \in I} A_i \iff x \in A_i, \text{ for all } i \in I \]

Note that

\[ \bigcap_{i \in I} A_i \subseteq A_i, \text{ for all } i \in I \]

the sets $A_i, i \in I$ are said to be **disjoint** if

\[ \bigcap_{i \in I} A_i = \emptyset \]
Brief Review of Set Theory - II

- **Set union**: given a family of sets \( A_i, \ i \in I \)

\[
x \in \bigcup_{i \in I} A_i \iff x \in A_i, \text{ for at least one } i \in I
\]

Note that

\[
A_i \subseteq \bigcup_{i \in I} A_i, \text{ for all } i \in I
\]

so that

\[
\bigcap_{i \in I} A_i \subseteq A_i \subseteq \bigcup_{i \in I} A_i, \text{ for all } i \in I
\]
Brief Review of Set Theory - III

- **Set Difference**: given two sets $A$ and $B$

  $$x \in A - B \iff x \in A \text{ and } x \notin B$$

  Note that

  $$A - B \subseteq A$$

  and

  $$(A - B) \cap B = \emptyset$$
Brief Review of Set Theory - IV

- **Set complement**: given a *universal set* $E$ and a set $A \subseteq E$

\[ x \in A^c \iff x \in E \text{ and } x \notin A \]

Clearly

\[ A^c = E - A \]

and

\[ A - B = A \cap B^c \]

De Morgan's Law: given sets $A_i$, $i \in I$

\[ \left( \bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c \]
Binary Images and Sets

• Assume a universal set $E = \mathbb{Z}^2$ (for discrete images) or $E = \mathbb{R}^2$ (for continuous images). For higher-dimensional "images", one has $E = \mathbb{Z}^d$ or $E=\mathbb{R}^d$ with $d>2$.

• Any given set $A \subseteq E$ corresponds to a binary image $I_A : E \rightarrow \{0,1\}$:

$$I_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in A^c \end{cases}$$

the binary image $I_A$ is sometimes called an indicator function.

• The converse is also true: any binary image $f : E \rightarrow \{0,1\}$ corresponds to a set $A_f \subseteq E$:

$$A_f = \{x \in E \mid f(x) = 1\}$$

this is called the image foreground, while its complement $A_f^c$ is the image background.
Logic and Order Operations

- The equivalence between sets and binary images leads to the realization that set operations are actually logic and order operations.

- Set intersection corresponds to **AND** and to **minimum**
  \[ I_{A \cap B}(x) = I_A(x) \text{ AND } I_B(x) = \min\{I_A(x), I_B(x)\} \]

- Set union corresponds to **OR** and to **maximum**
  \[ I_{A \cup B}(x) = I_A(x) \text{ OR } I_B(x) = \max\{I_A(x), I_B(x)\} \]

- Set complementation corresponds to **negation**
  \[ I_{A^c}(x) = \text{NEG } I_A(x) = 1 - I_A(x) \]
Translation and Reflection

• Since $E = \mathbb{Z}^2$ or $E = \mathbb{R}^2$ have an Abelian group structure for addition, one can define translation and reflection of a set.

• The translation of a set $B \subseteq E$ to point $x \in E$ is

$$B_z = \{x \in E \mid x = b + z, \text{ for } b \in B\}$$

• The reflection of $B \subseteq E$ is

$$\tilde{B} = \{x \in E \mid x = -b, \text{ for } b \in B\}$$

• $B$ is symmetric if

$$\tilde{B} = B$$

• Both translation and reflection are affected by the position of $B$ with respect to the origin $o \in E$. 
Structuring Element

- The basic operations in MM are defined in terms of a small set, called the structuring element (S.E.).

- The S.E. may be seen as a template used to probe an image. Both the shape and size of the S.E. affect the result of the operation. S.E.s are usually (but not always) symmetric and contain the origin.

- Examples (note the position of the origin, marked by a black dot):
  
  - 3x3 "cross"
  - 3x3 "box"
  - 5x1 "line"
  - 5x5 "cross"
Erosion

- It can be said that the most basic MM operator is the erosion.
- Given an image $A \subseteq E$ and a S.E. $B \subseteq E$, the erosion of $A$ by $B$ is

$$A \ominus B = \{z \in E \mid B_z \subseteq A\}$$

therefore, the erosion is equal to the points where one may "place" the S.E. such that it "fits" in the image.

- Alternatively, it can be shown that

$$A \ominus B = \bigcap_{b \in B} A_{-b}$$

- Provided that the origin $o$ is contained in $B$, erosion has the effect of shrinking the image, that is,

$$o \in B \Rightarrow A \ominus B \subseteq A$$
Example
Erosion as Morphological Filter

- The term "filter" has a precise meaning in MM, according to which erosion is not a filter. However, erosion is a (nonlinear) filter in the general sense of the word, as seen in the following example.
Properties of Erosion

\[ A_z \ominus B = (A \ominus B)_z \quad \text{translation-invariant} \]

\[ A_1 \subseteq A_2 \Rightarrow A_1 \ominus B \subseteq A_2 \ominus B \quad \text{increasing} \]

\[
\left( \bigcap_{i \in I} A_i \right) \ominus B = \bigcap_{i \in I} (A_i \ominus B) \quad \text{distributive over intersection}
\]

\[ A \ominus \{o\} = A \quad \text{origin is neutral element} \]

\[ B_1 \subseteq B_2 \Rightarrow A \ominus B_1 \supseteq A \ominus B_2 \quad \text{scaling property} \]
Dilation

- The "inverse" operation to erosion is dilation (but not a true inverse).
- Given an image $A \subseteq E$ and a S.E. $B \subseteq E$, the dilation of $A$ by $B$ is

$$A \oplus B = \{ z \in E | (\tilde{B})_z \cap A \neq \emptyset \}$$

therefore, the dilation is equal to the points where if one "places" the (reflected) S.E., it "hits" the image. Obviously, if $B$ is symmetric, one does not need to worry about reflecting $B$.

- Alternatively, it can be shown that

$$A \oplus B = \bigcup_{b \in B} A_b$$

- From this, and recalling that union is equivalent to maximum, it can be seen that dilation is a mask-based maximum filter (similar to the median filter). Similarly, erosion is a mask-based minimum filter.
Provided that the origin is contained in $B$, dilation has the effect of expanding the image, that is,

$$o \in B \Rightarrow A \subseteq A \oplus B$$

Dilation is commutative, that is,

$$A \oplus B = B \oplus A$$

From this and the definition on the previous page, we have

$$A \oplus B = \bigcup_{a \in A} B_a$$

that is, dilation corresponds to sliding the "mask" $B$ over all points of the original image and taking the union. This process is sometimes called morphological convolution.
Example
Dilation as Morphological Filter

• Once again, strictly speaking, dilation is not a "morphological filter," (more on MM filters shortly), but in a general sense, dilation performs nonliner filtering. For example:

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.
Properties of Dilation

\[ A_z \oplus B = (A \oplus B)_z \]  translation-invariant

\[ A_1 \subseteq A_2 \Rightarrow A_1 \oplus B \subseteq A_2 \oplus B \]  increasing

\[
\left( \bigcup_{i \in I} A_i \right) \oplus B = \bigcup_{i \in I} (A_i \oplus B) \]  distributive over union

\[ A \oplus B = B \oplus A \]  commutative

\[ (A \oplus B) \oplus C = A \oplus (B \oplus C) \]  associative

\[ A \oplus \{o\} = A \]  origin is neutral element

\[ \{o\} \oplus B = B \]  "impulse response"

\[ B_1 \subseteq B_2 \Rightarrow A \oplus B_1 \subseteq A \oplus B_2 \]  scaling property
Duality of Erosion and Dilation

- Erosion and dilation are not the exact inverses of each other, but for each given fixed S.E., they form pairs of dual operations.

- One form of such duality is called complementation duality. Given a fixed S.E. $B \subseteq E$, we obtain, using De Morgan's Law:

$$ (A \ominus B)^c = \left( \bigcap_{b \in B} A_{-b} \right)^c = \bigcup_{b \in B} A_{-b}^c = A^c \ominus \tilde{B} $$

Similarly, we can show

$$ (A \oplus B)^c = A^c \ominus \tilde{B} $$

- Assuming symmetric $B$ for simplicity, we can see that dilating an image is equivalent to eroding its background (and then taking the complement), and eroding an image is equivalent to dilating its background (and then taking the complement).
Opening

- Dilation is not the inverse of erosion; that is, if one composes them, one does not get identity. What does one get?

- There are two answers, depending on the order of composition.

- One of the things one gets is called opening. Given an image \( A \subseteq E \) and a S.E. \( B \subseteq E \), the opening of \( A \) by \( B \) is

\[
A \ominus B = (A \ominus B) \oplus B
\]

- Using the previous definitions of erosion and dilation, it can be easily shown that

\[
A \ominus B = \bigcup_{z \in E} \{B_z \mid B_z \subseteq A\}
\]
**Opening - II**

- Opening therefore is equal to sliding the S.E. (the "mask" B) and unionizing its translates at all points where it "fits" in the image (this is similar, but different than erosion).

- Opening can be described thus as "rolling a ball from the inside."
• Composition of erosion and dilation the other way is called **closing**. Given an image \( A \subseteq E \) and a S.E. \( B \subseteq E \), the closing of \( A \) by \( B \) is

\[
A \bullet B = (A \oplus B) \ominus B
\]

• Closing can be described as "rolling a ball from the outside."
Opening and Closing as MM Filters

• In addition to rounding convex corners, opening eliminates narrow "isthmuses" and "capes," and small "islands."

\[
A \triangledown B = (A \ominus B) \ominus B
\]

• In addition to rounding concave corners, closing fills up narrow "channels" and "inlets," and small "lakes."

\[
A \bullet B = (A \oplus B) \ominus B
\]
Properties of Opening

\[ A_z \circ B = (A \circ B)_z \] translation-invariant

\[ A_1 \subseteq A_2 \Rightarrow A_1 \circ B \subseteq A_2 \circ B \] increasing

\[ A \circ B = (A \circ B) \circ B \] idempotent

\[ A \circ B \subseteq A \] anti-extensive

\[ A \circ B_z = A \circ B \text{ for any } z \in E \] invariant to origin of S.E.

\[ A \circ \{z\} = A \circ \{o\} = A \text{ for any } z \in E \] any singleton is a neutral element
Properties of Closing

\[ A_z \bullet B = (A \bullet B)_z \quad \text{translation-invariant} \]

\[ A_1 \subseteq A_2 \Rightarrow A_1 \bullet B \subseteq A_2 \bullet B \quad \text{increasing} \]

\[ A \bullet B = (A \bullet B) \bullet B \quad \text{idempotent} \]

\[ A \subseteq A \bullet B \quad \text{extensive} \]

\[ A \bullet B_z = A \bullet B \quad \text{for any } z \in E \quad \text{invariant to origin of S.E.} \]

\[ A \bullet \{z\} = A \bullet \{o\} = A \quad \text{for any } z \in E \quad \text{any singleton is a neutral element} \]
Duality of Opening and Closing

• Similarly to erosion and dilation, opening and closing are dual by complementation.

• Given a fixed S.E. $B \subseteq E$, we obtain, by repeated application of the previous duality result for dilation and erosion

\[
(A \ominus B)^c = [(A \ominus B) \oplus B]^c = (A \ominus B)^c \ominus \tilde{B}
\]

\[
= (A^c \oplus \tilde{B}) \ominus \tilde{B} = A^c \bullet \tilde{B}
\]

Similarly, we can show

\[
(A \bullet B)^c = A^c \ominus \tilde{B}
\]

• Assuming symmetric $B$ for simplicity, we can see that opening an image is equivalent to closing its background (and then taking the complement), and closing an image is equivalent to opening its background (and then taking the complement).
Granulometries

• Given a S.E. \( B \subseteq E \), a set \( A \subseteq E \) is said to be \( B \)-open if

\[
A \circ B = A
\]

that is, \( A \) is invariant to opening by \( B \). It can be shown that this is true if and only if \( A = X \oplus B \) for some \( X \). It is also clear that \( A \circ B \) is \( B \)-open, by idempotence.

• The scaling property for openings is

\[
\begin{align*}
A \circ B_1 \subseteq A \circ B_2 \\
(A \circ B_1) \circ B_2 = (A \circ B_2) \circ B_1 = A \circ B_1
\end{align*}
\]

• For an integer \( k \geq 1 \), define the scaled S.E. \( kB \):

\[
kB = \underbrace{B \oplus B \oplus \cdots \oplus B}_k \text{ terms}
\]

It is clear that if \( r \leq k \), then \( kB \) is \( rB \)-open (why?).
Granulometries - II

• The family \( \{ A \ominus kB \mid k \geq 1 \} \) is called a granulometry.

• The term comes from the mineralogy field, where it corresponds to the classification of grainy material by means of sieving. Applying the granulometry of openings corresponds to sieving with varying sieve mesh size, because, from the scaling property, for \( r \leq k \),

\[
A \ominus kB \subseteq A \ominus rB
\]

\[
(A \ominus kB) \ominus rB = (A \ominus rB) \ominus kB = A \ominus kB
\]

• The first property says that more and more of the image is removed as \( k \) increases (i.e., a granulometry forms a pyramid representation). The second property means that applying sieves of different mesh size is the same as applying just the sieve of the largest size.

• The analogous development for families of closings of increasing size leads to the concept of anti-granulometries.
Granulometry Example

Printed circuit board (PCB) inspection

A

A \bigcirc 2B

A \bigcirc 4B

B
Counter-Example

- The definition of scaled replicas $kB$ of the S.E. is fundamental to guarantee the scaling property. Consider this example:

Note that $B_1 \subseteq B_2$ but $A \circ B_1 \not\subseteq A \circ B_2$.

This illustrates the fact that more than inclusion of $B_1$ into $B_2$ is required; one needs that $B_2$ be $B_1$-open.

Pattern Spectrum

• Given a granulometry \( \{A \ominus kB \mid k \geq 1\} \), the pattern spectrum is

\[
\text{PS}(k) = \text{Area}[A \ominus (k - 1)B - A \ominus kB], \quad k \geq 1
\]

where the area of a continuous set has the usual meaning, while the area of a discrete set is just its cardinality, i.e., its number of pixels, and where \( A \ominus 0B = A \ominus \{o\} = A \).

• The pattern spectrum thus measures the size of the details that are removed at step \( k \) of the granulometry. This has an analogy to the Fourier transform; rather than measuring the content at a given frequency, the PS measures the content at a given size.

• The pattern spectrum can also be defined for anti-granulometries (closings). Sometimes the two are combined, and the PS is defined for negative \( k \) (closing of size \( k \)) and positive \( k \) (opening of size \( k \)).
Pattern Spectrum Example

General Openings and Closings

• Any operator that is increasing, idempotent, and anti-extensive is said to be an opening.

• Similarly, any operator that is increasing, idempotent, and extensive is said to be a closing.

• Sometimes these operators are referred to as algebraic openings and algebraic closings.

• The opening and closing introduced earlier are sometimes called structural opening and structural closing. They are examples of translation-invariant algebraic openings and closings, respectively.

• An important example of algebraic opening is the area opening. This operator removes all connected components of area less than a given threshold $S$. Similarly, the area closing fills up all holes of area less than a given threshold $S$. 
Example of Area Opening

Automatic test scoring

binarized image of bubble answer sheet

area opening with S=500 extracts filled-in bubbles
Morphological Filters

• Even more generally, any operator that is increasing and idempotent is said to be a morphological filter, or simply a filter (when no confusion is possible).

• Algebraic openings form the class of anti-extensive filters, whereas algebraic closings form the class of extensive filters.

• Our familiar openings and closings, as well area openings and area closings, are morphological filters.

• According to this definition, erosions and dilations are not filters (they lack the idempotence property).
Alternating Sequential Filters

- It is possible to compose opening and closing, obtaining open-close and close-open filters

\[(A \circ B) \bullet B \quad \text{and} \quad (A \bullet B) \circ B\]

- Further composing gives close-open-close and open-close-open filters

\[((A \circ B) \bullet B) \circ B \quad \text{and} \quad ((A \bullet B) \circ B) \bullet B\]

- Further composition does not yield new operators, because

\[(({(A \circ B) \bullet B) \circ B}) \bullet B = (A \circ B) \bullet B\]

\[(({(A \bullet B) \circ B}) \bullet B) \circ B = (A \bullet B) \circ B\]

- Open-close, close-open, close-open-close and open-close-open filters, as well as their composition using different structuring elements, are called alternating sequential filters.
Example of Alternating Sequential Filter

\[(A \ominus B) \oplus B = A \circ B\]

\[(A \circ B) \oplus B = (A \circ B) \bullet B\]
Hit-or-Miss Transform

- Given an image $A \subseteq E$ and two S.E.s $B_1, B_2 \subseteq E$, the **hit-or-miss transform** of $A$ by $B_1, B_2$ is

  $$A \ominus (B_1, B_2) = \{ z \in E \mid (B_1)_z \subseteq A \text{ and } (B_2)_z \subseteq A^c \}$$

  therefore, the hit-or-miss transform is equal to the points where one may "place" the first S.E. such that it "fits" in the image foreground and at the same time place the second S.E. such that it fits in the image background.

- The hit-or-miss transform is another MM operation that can be defined in terms of simpler operations (in this case, erosion):

  $$A \ominus (B_1, B_2) = (A \ominus B_1) \cap (A^c \ominus B_2)$$

- This is clearly translation-invariant, and it is also true that

  $$A \ominus (B_1, B_2) = A^c \ominus (B_2, B_1)$$
Example

- The hit-or-miss transform can be used for shape detection.
- Example: detection of lower-left corners in an object.

Geodesic Dilation

• Given an image \( A \subseteq E \), a marker \( M \subseteq E \) and a S.E. \( B \subseteq E \), the geodesic dilation of size 1 of \( M \) by \( B \) conditioned on \( A \) is

\[
D_A^{(1)}(M) = (M \oplus B) \cap A
\]

the intersection limits the dilation of \( M \) to the interior of \( A \) (which is therefore sometimes called the mask). Note that we omit the S.E. \( B \) from the operator symbol, but it is always implicit in what follows.

• The geodesic dilation of size \( n \) can be defined recursively as

\[
D_A^{(n)}(M) = D_A^{(1)} \left[ D_A^{(n-1)}(M) \right]
\]

where \( D_A^{(0)}(M) = M \).
Geodesic Dilation - Example
The dual of geodesic dilation is called geodesic erosion.

Given an image \( A \subseteq E \), a marker \( M \subseteq E \) and a S.E. \( B \subseteq E \), the geodesic erosion of size 1 of \( M \) by \( B \) conditioned on \( A \) is

\[
E_A^{(1)}(M) = (M \ominus B) \cup A
\]

the union limits the erosion of \( M \) to the exterior of \( A \) (as before, \( A \) is called the mask but now it limits the operation "from below").

The geodesic erosion of size \( n \) can be defined recursively as

\[
E_A^{(n)}(M) = E_A^{(1)}\left[ E_A^{(n-1)}(M) \right]
\]

where \( E_A^{(0)}(M) = M \).
Geodesic Erosion - Example

Marker, $F$

Marker eroded by $B$

$B$

Geodesic erosion, $E_G^{(1)}(F)$

Mask, $G$
Morphological Reconstruction

• Geodesic operations lead to the concept of **morphological reconstruction**, a powerful tool in MM.

• Here, the S.E. itself ceases to be the main focus of interest, but rather it is the **connectivity** implied by the S.E. which is important.

• Given an image \( A \subseteq E \), a marker \( M \subseteq E \) and a S.E. \( B \subseteq E \), the **reconstruction** (by dilation) of \( M \) by \( B \) conditioned on \( A \) is

\[
R^D_A(M) = \lim_{n \to \infty} D^{(n)}_A(M) = \bigcup_{n=1}^{\infty} D^{(n)}_A(M)
\]

In practice, for finite images, the limit corresponds to iteration of the geodesic dilation until stability, that is,

\[
R^D_A(M) = D^{(k)}_A(M)
\]

where \( D^{(k)}_A(M) = D^{(k+1)}_A(M) \).
Morphological Reconstruction - Example

\[ D_G^{(1)}(F) \text{ dilated by } B \]
\[ D_G^{(2)}(F) \]
\[ D_G^{(2)}(F) \text{ dilated by } B \]
\[ D_G^{(3)}(F) \]
\[ D_G^{(3)}(F) \text{ dilated by } B \]
\[ D_G^{(4)}(F) \]
\[ D_G^{(4)}(F) \text{ dilated by } B \]
\[ D_G^{(5)}(F) = R_G^D(F) \]
Connected Component Extraction

- Reconstruction performs connected component extraction. The S.E. determines the connectivity.
- S.E. = 3x3 cross $\Rightarrow$ 4-connectivity.
- S.E. = 3x3 box $\Rightarrow$ 8-connectivity.
Dual Reconstruction

- Given an image $A \subseteq E$, a marker $M \subseteq E$ and a S.E. $B \subseteq E$, the **dual reconstruction** (by erosion) of $M$ by $B$ conditioned on $A$ is

$$R_A^E(M) = \lim_{n \to \infty} E_A^{(n)}(M) = \bigcap_{n=1}^{\infty} E_A^{(n)}(M)$$

As before, for finite images, the limit corresponds to iteration of the geodesic erosion until stability, that is,

$$R_A^E(M) = E_A^{(k)}(M)$$

where $E_A^{(k)}(M) = E_A^{(k+1)}(M)$.

- The dual reconstruction is indeed the dual of reconstruction:

$$(R_A^E(M))^c = R_{Ac}^D(M^c)$$

with the same implicit S.E. $B$ (assuming symmetric $B$, otherwise there is a reflection involved).
Close-Holes Filter

• Reconstruction can be used to fill up all holes in an image, where a "hole" is any connected component of the background that is not connected to the image border.

• Let the marker image be given by

\[ I_M(x, y) = \begin{cases} 
1 - I_A(x, y), & \text{if } (x, y) \text{ is on image border} \\
0, & \text{otherwise} 
\end{cases} \]

then \( H = \left[ R_{A^c}^D(M) \right]^c = R_A^E(M^c) \) is an image equal to \( A \) with all holes closed. This is really a dual reconstruction operation.
Border-Clearing Filter

- Similarly, if the marker image is now given by

\[
I_M(x, y) = \begin{cases} 
I_A(x, y), & \text{if } (x, y) \text{ is on image border} \\
0, & \text{otherwise}
\end{cases}
\]

then \( X = A - R_A^D(M) \) is an image equal to \( A \) with all components touching the border removed.

- Example: Text processing.
Opening by Reconstruction

• Given an image $A \subseteq E$ and a S.E. $C \subseteq E$, the opening by reconstruction of $A$ by $C$ (for a given connectivity) is

$$A \ominus_R C = R^D_A(A \ominus C)$$

This is simply reconstruction where the marker is obtained by a structural opening of the original image by the S.E. $C$. We deliberately use the letter "C" to make clear this is distinct from the underlying S.E. $B$ that gives the connectivity.

• Example: Text processing. The aim is to extract characters containing long vertical strokes. To obtain this, $C$ must be a vertical line S.E.
Opening by Reconstruction - II

- The opening by reconstruction has exactly the same properties of the structural opening: it is increasing, idempotent, anti-extensive, insensitive to origin of S.E. $C$, and any singleton is a neutral element.

- The family $\{A \circ_{R} kB \mid k \geq 1\}$ is a granulometry by reconstruction. It has all the properties of a regular granulometry, including, for $r \leq k$,

$$A \circ_{R} kB \subseteq A \circ_{R} rB$$

$$(A \circ_{R} kB) \circ_{R} rB = (A \circ_{R} rB) \circ_{R} kB = A \circ_{R} kB$$

The S.E. $B$ is often, but not necessarily, the same S.E. used for the connectivity. One may also define a pattern spectrum just as before.

- The dual operator is the closing by reconstruction:

$$A \bullet_{R} C = R_{A}^{E}(A \bullet C)$$

This operator has exactly the same properties of the structural closing. It can also be used to define anti-granulometries.
Granulometry by Reconstruction Example

Printed circuit board (PCB) inspection

$A$, $A \circ R 4B$, $A \circ R 6B$

$B$
Boundary Extraction

Given an image $A \subseteq E$ and a symmetric S.E. $B \subseteq E$, boundary extraction can be accomplished in MM in several ways:

- **Interior boundary:**
  \[ \beta_1(A) = A - A \ominus B \]

- **Exterior boundary:**
  \[ \beta_2(A) = A \oplus B - A \]

- **Centered boundary:**
  \[ \beta_3(A) = A \oplus B - A \ominus B \]
Boundary Extraction - Example

interior boundary

![Diagram showing interior boundary with a black dot labeled B]
A skeleton is a reduced representation that contains the major structural features of a binary image. It should be possible to invert the process and recover the image from its skeleton.

For continuous shapes, the skeleton corresponds to the medial axis.
Discrete Skeleton

• A simple MM algorithm for computing the skeleton of a discrete image is based on successive erosions and opening.

• Consider the skeleton subsets:

\[ S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B, \quad k = 0, 1, \ldots \]

• Note: using the following property of erosion

\[ (A \ominus B_1) \ominus B_2 = A \ominus (B_1 \oplus B_2) \]

we can write

\[ A \ominus kB = (\cdots ((A \ominus B) \ominus B) \ominus \cdots) \ominus B \]

so that the skeleton subsets are found by means of using successive erosions and an opening at each step.
Discrete Skeleton - II

• The discrete morphological skeleton of $A$ is given by

$$S(A) = \bigcup_{k=0}^{\infty} S_k(A)$$

In practice, the skeleton subsets become empty after a certain number of steps, so the above set union operation is finite.

• It can be shown that the image can be recovered by

$$A = \bigcup_{k=0}^{\infty} S_k(A) \oplus kB$$

As before, this is in actuality a finite union operation.
### Discrete Skeleton - Example

<table>
<thead>
<tr>
<th>$k$</th>
<th>$A \ominus kB$</th>
<th>$(A \ominus kB) \circ B$</th>
<th>$S_k(A)$</th>
<th>$\bigcup_{k=0}^{K} S_k(A)$</th>
<th>$S_k(A) \ominus kB$</th>
<th>$\bigcup_{k=0}^{K} S_k(A) \ominus kB$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
</tr>
<tr>
<td>1</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
</tr>
</tbody>
</table>

**Legend:**
- $A$ and $B$ are regions in a digital image.
- $\ominus$ denotes morphological erosion.
- $\circ$ denotes morphological opening.
- $S_k(A)$ represents the $k$-th skeleton of region $A$.
- $\bigcup_{k=0}^{K} S_k(A)$ represents the union of all skeletons up to $k=K$. 
Grayscale Morphology

• There are several approaches to extend MM to grayscale images.

• **Umbra** approach: if an image is thought of as a surface, the volume under the image is a set, called the "umbra." One can apply binary MM operations to this and then reconstruct a grayscale image.

• **Threshold decomposition** approach: any image can be decomposed in a series of binary images by thresholding. One can apply binary MM operations to these and then reconstruct a grayscale image.

• **Algebraic** approach: one can use the equivalence between set operations and general order operations (e.g., union and intersection are equivalent to max and min, respectively). The latter can be applied to grayscale images. This is the approach we will follow.
**Grayscale Images**

- Similarly as before, we assume an image domain \( E = \mathbb{Z}^d \) (for discrete images) or \( E = \mathbb{R}^d \) (for continuous images). For higher-dimensional "images", one has \( E = \mathbb{Z}^d \) or \( E = \mathbb{R}^d \) with \( d > 2 \).

- A grayscale image is a function \( f: E \rightarrow \{0, 1, \ldots, N-1\} \) where \( N \) is the number of grayvalues. For example, for an 8-bit image, \( N = 256 \).
"Inclusion" of two grayscale images is defined by the pointwise order relation:

\[ f \leq g \iff f(x) \leq g(x) \]
"Intersection" corresponds to the infimum $\bigwedge f_i$, which is given by the pointwise minimum:

$$\left( \bigwedge_{i \in I} f_i \right)(x) = \min\{f_i(x) \mid i \in I\}$$
Image Supremum

- "Union" corresponds to the supremum $\bigvee f_i$, which is given by the pointwise maximum:

$$\left( \bigvee_{i \in I} f_i \right)(x) = \max\{f_i(x) \mid i \in I\}$$
Image Difference and Negation

- "Set difference" \( f - g \) is defined by pointwise subtraction:
  \[
  (f - g)(x) = f(x) - g(x)
  \]

- "Set complement" is defined by pointwise negation:
  \[
  f^c(x) = (N - 1) - f(x)
  \]
Grayscale Erosion

- Given an image $f: E \rightarrow \{0, 1, \ldots, N-1\}$ and a S.E. $B \subseteq E$, the flat erosion of $f$ by $B$ is

$$
(f \ominus B)(x) = \bigwedge_{b \in B} f(x + b)
= \min \{f(x + b) \mid b \in B\}
$$

- Therefore, the grayscale flat erosion is a min filter. It can still be seen as fitting the S.E., this time the fitting occurs "under" the image.

- Provided that the origin $o$ is contained in $B$, erosion has the effect of darkening the image, that is,

$$
o \in B \Rightarrow f \ominus B \leq f
$$
Grayscale Dilation

- Given an image \( f: E \rightarrow \{0, 1, ..., N-1\} \) and a S.E. \( B \subseteq E \), the \textbf{flat dilation} of \( f \) by \( B \) is

\[
(f \oplus B)(x) = \bigvee_{b \in B} f(x - b)
\]

\[
= \max \{ f(x - b) \mid b \in B \}
\]

due to, the grayscale flat dilation is a max filter. It can still be seen as the loci of points where the (reflected) S.E. "hits" the image.

- Provided that the origin is contained in \( B \), dilation has the effect of \textbf{brightening} the image, that is,

\[
o \in B \Rightarrow f \leq f \oplus B
\]
Example

original image  erosion by disk of radius 2  dilation by disk of radius 2
Properties of Grayscale Erosion

\[ f(x - a) \ominus B = (f \ominus B)(x - a) \quad \text{translation-invariant} \]

\[ f_1 \leq f_2 \Rightarrow f_1 \ominus B \leq f_2 \ominus B \quad \text{increasing} \]

\[ \left( \bigwedge_{i \in I} f_i \right) \ominus B = \bigwedge_{i \in I} (f_i \ominus B) \quad \text{distributive over infimum} \]

\[ f \ominus \{o\} = f \quad \text{origin is neutral element} \]

\[ B_1 \subseteq B_2 \Rightarrow f \ominus B_1 \geq f \ominus B_2 \quad \text{scaling property} \]
Properties of Grayscale Dilation

\[ f(x-a) \oplus B = (f \oplus B)(x-a) \quad \text{translation-invariant} \]

\[ f_1 \leq f_2 \Rightarrow f_1 \oplus B \leq f_2 \oplus B \quad \text{increasing} \]

\[ \left( \bigvee_{i \in I} f_i \right) \oplus B = \bigvee_{i \in I} (f_i \oplus B) \quad \text{distributive over supremum} \]

\[ (f \oplus B) \oplus C = f \oplus (B \oplus C) \quad \text{associative} \]

\[ f \oplus \{o\} = f \quad \text{origin is neutral element} \]

\[ B_1 \subseteq B_2 \Rightarrow f \oplus B_1 \leq f \oplus B_2 \quad \text{scaling property} \]
Duality of Grayscale Erosion and Dilation

- Just as in the binary case, grayscale erosion and dilation are not the exact inverses of each other, but for each given fixed S.E., they form pairs of dual operations.

- As we saw before, one form of such duality is called complementation duality. Given a fixed S.E. $B \subseteq E$, it can be shown that

\[
(f \ominus B)^c = f^c \ominus \bar{B}
\]

and

\[
(f \oplus B)^c = f^c \ominus \bar{B}
\]

- Assuming symmetric $B$ for simplicity, we can see that dilating an image is equivalent to eroding its negative (complement) and then taking the complement, and eroding an image is equivalent to dilating its negative, and then taking the complement.
Grayscale Morphological Gradient

- Corresponding to boundary extraction in binary MM, we have the morphological gradient in grayscale MM.
Nonflat Grayscale Morphology

- It is also possible to define erosion and dilation between grayscale images (that is, the S.E. itself is a grayscale image).

- Given two images $f, g: E \rightarrow \{0, 1, \ldots, N-1\}$, the nonflat erosion and nonflat dilation of $f$ by $g$ are given by

$$
(f \ominus g)(x) = \max \{ f(x - a) + g(a) \mid a \in E \}
$$

$$
(f \oplus g)(x) = \min \{ f(x + a) - g(a) \mid a \in E \}
$$

- A Nonflat S.E. is usually an image with a small domain of definition. Nonflat MM operators are not commonly used in image processing.
Grayscale Opening and Closing

• Given an image \( f : E \rightarrow \{0, 1, \ldots, N-1\} \) and a S.E. \( B \subseteq E \), the flat opening and flat closing of \( f \) by \( B \) is obtained just as in the binary case:

\[
f \circ B = (f \ominus B) \oplus B \quad \quad f \bullet B = (f \oplus B) \ominus B
\]

• Opening and closing still have the interpretation of "stamping" the S.E., "under" the image and "above" the image, respectively.
Example

original image  opening by disk of radius 3  closing by disk of radius 5
Properties of Grayscale Opening

\[ f(x - a) \circ B = (f \circ B)(x - a) \quad \text{translation-invariant} \]

\[ f_1 \leq f_2 \Rightarrow f_1 \circ B \leq f_2 \circ B \quad \text{increasing} \]

\[ f \circ B = (f \circ B) \circ B \quad \text{idempotent} \]

\[ f \circ B \leq f \quad \text{anti-extensive} \]

\[ f \circ B_z = f \circ B \quad \text{for any } z \in E \quad \text{invariant to origin of S.E.} \]

\[ f \circ \{z\} = f \circ \{o\} = f \quad \text{for any } z \in E \quad \text{any singleton is a neutral element} \]
Properties of Grayscale Closing

\[ f(x - a) \ast B = (f \ast B)(x - a) \quad \text{translation-invariant} \]

\[ f_1 \leq f_2 \Rightarrow f_1 \ast B \leq f_2 \ast B \quad \text{increasing} \]

\[ f \ast B = (f \ast B) \ast B \quad \text{idempotent} \]

\[ f \leq f \ast B \quad \text{extensive} \]

\[ f \ast B_z = f \ast B \quad \text{for any } z \in E \quad \text{invariant to origin of S.E.} \]

\[ f \ast \{z\} = f \ast \{o\} = f \quad \text{for any } z \in E \quad \text{any singleton is a neutral element} \]
Duality of Grayscale Opening and Closing

• Just as in the binary case, opening and closing are dual by complementation.

• Given a fixed S.E. $B \subseteq E$, it can be shown that

$$ (f \circ B)^c = f^c \bullet \tilde{B} $$

and

$$ (f \bullet B)^c = f^c \circ \tilde{B} $$

• Assuming symmetric $B$ for simplicity, we can see that opening an image is equivalent to closing its negative (and then taking the complement), and closing an image is equivalent to opening its negative (and then taking the complement).
Grayscale Alternating Sequential Filters

- Just as in the binary case, it is possible to define grayscale ASFs for image smoothing.
The top-hat and bottom-hat operations are defined respectively as

\[ T_{\text{hat}}(f) = f - f \circ B \quad \text{and} \quad B_{\text{hat}}(f) = f \bullet B - f \]

A useful application of these operations is background removal.

![Original image](image1)
![Top-hat image](image2)

opening by disk of radius 40

A useful application of these operations is background removal.

![Original image](image1)
![Top-hat image](image2)

opening by disk of radius 40
Another interesting application of opening and closings is in the segmentation of textures.
Grayscale Granulometries

- The definition of $B$-open and the scaling property apply without change to grayscale opening.

- The family $\{f \circ kB \mid k \geq 1\}$ is called a grayscale granulometry.

- As in the binary case, the scaling property implies that, for $r \leq k$,

$$f \circ kB \leq f \circ rB$$

$$(f \circ kB) \circ rB = (f \circ rB) \circ kB = f \circ kB$$

- The analogous development for families of grayscale closings of increasing size leads to the concept of grayscale anti-granulometries.

- The concept of grayscale pattern spectrum can be defined as before. The "area" of the image is usually taken to be the volume under the image, considered as a surface.
Grayscale Granulometry Example

original image

\[ f = \text{open-close by } 5B \]

\[ f \circ 10B \]

\[ f \circ 20B \]

\[ f \circ 25B \]

\[ f \circ 30B \]

pattern spectrum
Grayscale Geodesic Dilation and Erosion

- Given a mask $g: E \rightarrow \{0, 1, \ldots, N-1\}$, a marker $f: E \rightarrow \{0, 1, \ldots, N-1\}$, and a S.E. $B \subseteq E$, the grayscale geodesic dilation and grayscale geodesic erosion of size 1 of $f$ by $B$ conditioned on $g$ are given by

$$D_g^{(1)}(f) = (f \oplus B) \land g$$
$$E_g^{(1)}(f) = (f \ominus B) \lor g$$

We omit the S.E. $B$ from the operator symbol, but it is always implicit.

- The grayscale geodesic dilation and erosion of size $n$ can be defined recursively as

$$D_g^{(n)}(f) = D_g^{(1)} [D_g^{(n-1)}(f)]$$
$$E_g^{(n)}(f) = E_g^{(1)} [E_g^{(n-1)}(f)]$$

where $D_g^{(0)}(f) = f$ and $E_g^{(0)}(f) = f$. 

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Grayscale Reconstruction

- Grayscale geodesic operations lead to grayscale reconstruction, a powerful and useful operation in a variety of imaging applications.

- As in binary case, the connectivity implied by the S.E. is the focus.

- Given a mask $g: E \rightarrow \{0,1,...,N-1\}$, a marker $f: E \rightarrow \{0,1,...,N-1\}$, and a S.E. $B \subseteq E$, the grayscale reconstruction (by dilation) of $f$ by $B$ conditioned on $g$ is given by

$$R_g^D(f) = \lim_{n \rightarrow \infty} D_g^{(n)}(f) = \bigvee_{n=1}^{\infty} D_g^{(n)}(f)$$

In practice, for finite images, the limit corresponds to iteration of the geodesic dilation until stability, that is,

$$R_g^D(f) = D_g^{(k)}(f)$$

where $D_g^{(k)}(f) = D_g^{(k+1)}(f)$. 
Grayscale Dual Reconstruction

• Given a mask $g: E \to \{0, 1, \ldots, N-1\}$, a marker $f: E \to \{0, 1, \ldots, N-1\}$, and a S.E. $B \subseteq E$, the grayscale dual reconstruction (by erosion) of $f$ by $B$ conditioned on $g$ is

$$R_g^E(f) = \lim_{n \to \infty} E_g^{(n)}(f) = \bigwedge_{n=1}^{\infty} E_g^{(n)}(f)$$

As before, for finite images, the limit corresponds to iteration of the geodesic erosion until stability, that is,

$$R_g^E(f) = E_g^{(k)}(f)$$

where $E_g^{(k)}(f) = E_g^{(k+1)}(f)$.

• The dual reconstruction is the dual of reconstruction:

$$(R_g^E(f))^c = R_{g^c}^D(f^c)$$

with same S.E. $B$ (assuming symmetric $B$, else there is a reflection).
Grayscale Opening/Closing by Reconstruction

- Given an image \( f: E \rightarrow \{0, 1, \ldots, N-1\} \) and a S.E. \( C \subseteq E \), the grayscale opening by reconstruction of \( A \) by \( C \) (for a given connectivity) is

\[
f \circ_{R} C = R^{D}_{f}(f \circ C)
\]

We deliberately use the letter "C" to make clear this is distinct from the underlying S.E. \( B \) that gives the connectivity.

- The grayscale opening by reconstruction has exactly the same properties of the structural grayscale opening.

- The family \( \{f \circ_{R} kB \mid k \geq 1\} \) is a granulometry by reconstruction.

- The dual operator is the closing by reconstruction:

\[
f \bullet_{R} C = R^{E}_{f}(f \bullet C)
\]

This operator has exactly the same properties of the structural grayscale closing. It can also be used to define anti-granulometries.
Example

original image  opening by rec with horiz line  top-hat by rec (*)  opening by rec with small horiz line

dilation with small horiz line (**)  minimum of (**) with (*)  reconstruction with minimum as marker and (**) as mask