Constructing Multiscale Connectivities

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Abstract

In a recent paper, we have proposed a novel theory that extends connectivity to a multiscale framework. Although we have presented several examples, we have not provided a systematic approach for constructing multiscale connectivities. In this paper, we introduce three general techniques for constructing multiscale connectivities. The first two start from a given base connectivity and produce multiscale connectivities by applying pyramids of clusterings or granulometries. The third technique starts from a given multiscale connectivity and constructs a new one by means of a clustering or an opening. Several examples illustrate the utility of the proposed techniques for constructing multiscale connectivities.

1 Introduction

In a previous paper [3], we introduced a novel theory of connectivity that considers traditional concepts in a multiscale framework (see also [1, 5, 18]). We have called this theory *multiscale connectivity.* The main motivation behind developing a theory for multiscale connectivity is the observation that connectivity may depend on the scale that an object is observed.

In [3], we have investigated several aspects of the theory of multiscale connectivity and presented several examples. However, we have not provided a systematic approach for constructing multiscale connectivities. It so happens that there are several examples of morphological operations that have a natural multiscale interpretation, such as dilations by scalable structuring elements, granulometries and anti-granulometries [6, 11]. The main purpose of this paper is to systematically define and study examples of multiscale connectivities generated by such operations.

More specifically, we investigate the use of three general techniques for constructing multiscale connectivities. The first two use *pyramids of clusterings* and *granulometries*, respectively, to construct a multiscale connectivity, starting from a given plain connectivity. This construction builds on ideas first proposed in [9, 12], and extends our construction methodology in [2] to the multiscale case. Multiscale connectivities generated by pyramids of clusterings are of a "negative" or clustering nature, whereas those generated by granulometries are of a "positive" or fragmenting nature. We present several examples based on pyramids of dilations and closings, and on structural granulometries, including an example for grayscale images, and an example based on morphological sampling operators (which was originally suggested to us by H. Heijmans — see also [2]). Moreover, we present a technique that produces new multiscale connectivities from given ones by means of a clustering or an opening. The resulting connectivities are referred to as *second-generation* multiscale connectivities, following the terminology used by J. Serra in the single-scale case [13].

Our development require that the reader recalls several mathematical concepts from the theory of complete lattices, mathematical morphology, and the theory of connectivity classes. The reader is referred to Section 2 of [2, 3], for a comprehensive review of these concepts. Following our nomenclature in [3], whenever we use the term "lattice" we mean "complete lattice."

The paper is organized as follows. In Section 2, we provide a brief overview of multiscale connectivity. The purpose of this section is to provide a link between the theory developed in [3] and the material presented in this paper. For more details on multiscale connectivity, the reader is referred to [3]. In Section 3 and 4, we construct multiscale connectivities by means of pyramids of clusterings and granulometries, respectively. In Section 5, we discuss second-generation multiscale connectivities. Finally, our conclusions are summarized in Section 6.

2 Multiscale Connectivity

This section provides a brief overview of the theory of multiscale connectivity. For more details, including proofs, the reader is referred to [3].

In what follows, Σ denotes a *scale lattice*. The least and greatest elements of Σ are denoted by O_Σ and I_Σ, respectively. Throughout this paper, $\Sigma_0 = \Sigma \setminus \{O_\Sigma\}$ and $\Sigma_1 = \Sigma \setminus \{I_\Sigma\}$. Two examples of scale lattices are of particular interest: the case when $\Sigma = \overline{\mathbb{R}}$, or a closed interval of $\overline{\mathbb{R}}$, and the case when the scale lattice is a finite chain, e.g., $\mathbb{K} = \{0, 1, \ldots, K\}$.

Given a lattice $\mathcal{L}$, furnished with a sup-generating family $\mathcal{S}$, we define a function $\varphi\colon\ \mathcal{L} \to \Sigma$ to be a *connectivity measure* on $\mathcal{L}$ if: (i) $\varphi(O) = \varphi(x) = I_\Sigma$, for $x \in \mathcal{S}$, and (ii) for a family $\{A_\alpha\}$ in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$, we have that $\varphi(\bigvee A_\alpha) \geq \bigwedge \varphi(A_\alpha)$. Given $A \in \mathcal{L}$, the quantity $\varphi(A)$ indicates the degree of connectivity of A. If $\varphi(A) = I_\Sigma$, A is said to be *fully connected*, whereas if $\varphi(A) = O_\Sigma$, A is said to be *fully disconnected*. Intermediate connectivity is defined by saying that A is *σ-connected* if $\varphi(A) \geq \sigma$, for some $\sigma \in \Sigma$. A connectivity measure φ on $\mathcal{L}$ is said to be *strong* if the greatest element I of $\mathcal{L}$ is fully connected; i.e., if $\varphi(I) = I_\Sigma$. If $\mathcal{L} = \mathcal{P}(E)$, with $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$, then φ is said to be *translation-invariant* if $\varphi(A) = \varphi(A_h)$, for all $h \in E$, where $A_h = \{a + h \mid a \in A\}$.

A concept intimately related to connectivity measures is that of a connectivity pyramid. A mapping $\mathbf{C}\colon\ \Sigma_0 \to \mathcal{P}(\mathcal{L})$ is a *connectivity pyramid* on $\mathcal{L}$ if: (i) $\mathbf{C}(\sigma)$ is a connectivity class in $\mathcal{L}$, for each $\sigma \in \Sigma_0$, (ii) $\mathbf{C}(\sigma) \subseteq \mathbf{C}(\tau)$, for $\sigma \geq \tau$, and (iii) $\mathbf{C}(\bigvee \sigma_\alpha) = \bigcap \mathbf{C}(\sigma_\alpha)$, for all nonempty set of scales $\{\sigma_\alpha\} \subseteq \Sigma_0$. If $\Sigma = \overline{\mathbb{R}}$, then condition (iii) can be verified to be equivalent to: (iii') $\mathbf{C}(\sigma) = \bigcap_{\tau<\sigma} \mathbf{C}(\tau)$, for all $\sigma \in \overline{\mathbb{R}}_0$. On the other hand, if $\Sigma = \mathbb{K}$, then condition (iii) becomes equivalent to condition (ii); hence, only conditions (i) and (ii) are required in the definition of discrete-scale connectivity pyramids. The connectivity class $\mathbf{C}(\sigma)$ is said to be the *σ-level* or the *σ-connectivity class* of $\mathbf{C}$; it may be thought of as the connectivity class assigned at scale σ. For $A \in \mathcal{L}$, if A is connected at all scales, i.e., if $A \in \mathbf{C}(I_\Sigma)$, then A is said to be fully connected, whereas if A is not connected at any scale, i.e., if $A \notin \bigcup_{\sigma\in\Sigma_0} \mathbf{C}(\sigma)$, then A is said to be fully disconnected. Intermediate connectivity is defined by saying that A is *σ-connected* if $A \in \mathbf{C}(\sigma)$, for some $\sigma \in \Sigma_0$. A connectivity pyramid $\mathbf{C}$ on $\mathcal{L}$ is said to be *strong* if $I \in \mathbf{C}(I_\Sigma)$ (so that I is fully connected); this is equivalent to each σ-connectivity class being strong. In addition, if $\mathcal{L} = \mathcal{P}(E)$, with $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$, we say that $\mathbf{C}$ is *translation-invariant* if each σ-connectivity class is translation-invariant; i.e., if $A \in \mathbf{C}(\sigma) \;\Leftrightarrow\; A_h \in \mathbf{C}(\sigma)$, for all $h \in E$, $\sigma \in \Sigma_0$.

For a given lattice $\mathcal{L}$, it can be shown that the set $\mathcal{M}(\mathcal{L})$ of all connectivity measures and the set $\mathcal{Y}(\mathcal{L})$ of all connectivity pyramids are complete lattices. Moreover, it can be shown that there is an isomorphism between lattices $\mathcal{M}(\mathcal{L})$ and $\mathcal{Y}(\mathcal{L})$. This is a bijection — to each connectivity measure φ on $\mathcal{L}$, there is an associated equivalent connectivity pyramid $\mathbf{C}$ on $\mathcal{L}$, given by

$$\Gamma(\varphi)(\sigma) = \mathbf{C}(\sigma) = \{A \in \mathcal{L} \mid \varphi(A) \geq \sigma\}, \quad \sigma \in \Sigma_0. \tag{1}$$

Conversely, φ can be regenerated by "stacking up" the σ-levels of $\mathbf{C}$, in which case

$$\Gamma^{-1}(\mathbf{C})(A) = \varphi(A) = \bigvee\{\sigma \in \Sigma_0 \mid A \in \mathbf{C}(\sigma)\}, \quad A \in \mathcal{L}. \tag{2}$$

Hence, a multiscale connectivity on $\mathcal{L}$ can be equivalently specified by either method.

The concept of connectivity measure is therefore equivalent to the concept of connectivity pyramid. Depending on the circumstances, one method may be more convenient than the other. Hence, we often say that $\mathcal{L}$ is furnished with a *multiscale connectivity system* $(\varphi, \mathbf{C}) \in \mathcal{M}(\mathcal{L}) \times \mathcal{Y}(\mathcal{L})$, such that φ and $\mathbf{C}$ are equivalent under the aforementioned bijection (see Theorem 3.14 in [3]).

Given a multiscale connectivity system $(\varphi, \mathbf{C})$ on a lattice $\mathcal{L}$, the *σ-connectivity openings* associated with $(\varphi, \mathbf{C})$ are given by

$$\gamma_x^\sigma(A) = \bigvee\{C \in \mathbf{C}(\sigma) \mid x \leq C \leq A\}, \quad A \in \mathcal{L},\ \sigma \in \Sigma_0,\ x \in \mathcal{S}.$$

It can be shown that σ-connectivity openings characterize, in a unique fashion, the multiscale connectivity systems with which they are associated (see Theorem 4.1 in [3]).

The decomposition of an element $A \in \mathcal{L}$ into its σ-connected components by means of a mapping $\mathbf{c}_A\colon \Sigma_0 \times \mathcal{S}(A) \to \mathcal{L}$, given by $\mathbf{c}_A(\sigma, x) = \gamma_x^\sigma(A)$, for $\sigma \in \Sigma_0$ and $x \in \mathcal{S}(A)$, where $\mathcal{S}(A) = \{x \in \mathcal{S} \mid x \leq A\}$, is called the *hierarchical partition of connected components* (HPCC) of A. For each $\sigma \in \Sigma_0$, $\mathbf{c}_A(\sigma, \cdot)$ is called the *σ-level* or the *σ-partition* of the HPCC $\mathbf{c}_A$ of A.

Given a marker $M \in \mathcal{L}$, the *σ-reconstruction* $\rho^\sigma(A \mid M)$ of $A \in \mathcal{L}$ from M is given by

$$\rho^\sigma(A \mid M) \;=\; \bigvee_{x \in \mathcal{S}(M)} \gamma_x^\sigma(A) \;=\; \bigvee\{C \in \mathcal{C}^\sigma(A) \mid C \wedge M \neq O\}, \quad \sigma \in \Sigma_0.$$

where $\mathcal{C}^\sigma(A)$ denotes the family of σ-connected components of A. Hence, the σ-reconstruction operator $\rho^\sigma(A \mid M)$ extracts the σ-connected components of A that "overlap" with marker M. It can be shown that, when $\mathcal{L}$ is infinite $\vee$-distributive and Σ is discrete, σ-reconstruction operators characterize, in a unique fashion, the multiscale connectivity systems with which they are associated, by means of the previous two equations (see Theorem 4.3 in [3]).

3 Multiscale Connectivities Based on Pyramids of Clusterings

In this section, we show how pyramids of clusterings can be used to construct multiscale connectivities. One feature of such connectivity is that, if an element is connected with respect to the base connectivity, then its degree of connectivity is zero and the element is fully connected. Otherwise, its degree of connectivity is negative. The more negative the degree of connectivity is, the "more disconnected" the element is with respect to the base connectivity.

3.1 Basic Notions

In this subsection, we give the basic definitions and results concerning multiscale connectivities based on pyramids of clusterings. In the next subsection, we proceed to give several examples that illustrate the generality of the approach. The following definition of an operator pyramid is due to [16].

3.1 Definition. For a given indexing poset J, a family $\{\psi_\alpha \mid \alpha \in J\}$ of operators on a lattice $\mathcal{L}$ is said to be a *pyramid* if:

$$\text{for } \alpha, \beta \in J, \text{ with } \alpha \geq \beta, \text{ there exists a } \gamma \in J \text{ such that } \psi_\alpha = \psi_\gamma \psi_\beta. \tag{3}$$

In other words, there is always an operator in the family that provides the necessary composition to move from a low indexed operator in the family to higher indexed one. The well-known example of operators that form pyramids are dilations and closings [16] (see Section 3.2 for more details on such pyramids).

In [12, 14], J. Serra constructed "second-generation" connectivity classes by taking the inverse mapping $\psi^{-1}(\mathcal{C})$ of a given base connectivity class $\mathcal{C}$, for certain kinds of operators ψ. This framework was later generalized in [2], using the notion of clustering. Below, we will see how this can exploited to construct multiscale connectivities.

The following definition of clustering appears in [2, 5].

3.2 Definition. An operator ψ on a lattice $\mathcal{L}$ is said to be a *clustering* with respect to a connectivity class $\mathcal{C}$ in $\mathcal{L}$ if

(*i*) ψ is increasing and extensive.

(*ii*) ψ is *connectivity-preserving*; i.e., $\psi(\mathcal{C}) \subseteq \mathcal{C}$.

(*iii*) For a family $\{A_\alpha\}$ in $\mathcal{L}$ such that $\psi(A_\alpha) \in \mathcal{C}$, $\forall\, \alpha$, and $\bigwedge A_\alpha \neq O$, we have that $\psi(\bigvee A_\alpha) \in \mathcal{C}$.

In addition, a *strong clustering* is an increasing and extensive operator ψ on $\mathcal{L}$ such that $\psi(O) = O$ and $\psi(\mathbf{id} \wedge \gamma_x \psi) = \gamma_x \psi$, $\forall\, x \in \mathcal{S}$, where **id** denotes the identity operator and $\{\gamma_x, x \in \mathcal{S}\}$ are the connectivity openings associated with $\mathcal{C}$. A strong clustering is a clustering, but not vice-versa [2, Prop.6 .2]. It will become clear below that he strong property is desirable because it leads to an explicit characterization of the σ-connectivity openings and σ-reconstruction operator of the generated multiscale connectivity.

The concept of clustering unify two well-known examples of operators used for constructing second-generation connectivities, originally proposed by J. Serra in [12, 14].

3.3 Proposition. Let $\mathcal{L}$ be a lattice furnished with a connectivity class $\mathcal{C}$. Connectivity-preserving extensive dilations and connectivity-preserving closings are clusterings on $\mathcal{L}$. Moreover, if $\mathcal{L}$ is infinite $\vee$-distributive, the strong clustering property holds in the case of dilations as well.

For a proof of this result, see Prop. 3.3 and [2, Prop. 6.8]). A useful fact, which will be exploited in Section 3.2, is that to establish connectivity-preservation for a dilation δ it is sufficient to show that it preserves connectivity of the sup-generators; i.e., that $\delta(\mathcal{C}) \subseteq \mathcal{C} \;\Leftrightarrow\; \delta(\mathcal{S}) \subseteq \mathcal{S}$.

Now, consider a pyramid of clusterings $\{\psi_\sigma \mid \sigma \in \Sigma_1\}$. We use $\Sigma_1 = \Sigma \setminus \{I_\Sigma\}$ instead of Σ since it will become clear by Prop. 3.6 that the top scale I_Σ is not needed for the construction of the multiscale connectivities based on pyramids of clusterings. We say that $A \in \mathcal{L}$ is a *σ-cluster* if $\psi_\sigma(A) \in \mathcal{C}$, for $\sigma \in \Sigma_1$. Note that, from the extensivity of clusterings and the pyramid property (3), it easily follows that $\psi_\sigma \leq \psi_\tau$, for $\sigma \leq \tau$, so that the operators in a pyramid of clusterings form a chain. Below, we show that pyramids of clusterings can be used to construct multiscale connectivities. First, we need the following definition.

3.4 Definition. A pyramid of clusterings $\{\psi_\sigma \mid \sigma \in \Sigma_1\}$ is said to be *coercive* if:

$$\psi_{\sigma_\alpha}(A) \in \mathcal{C},\ \forall \sigma_\alpha \;\Leftrightarrow\; \psi_{\wedge\sigma_\alpha}(A) \in \mathcal{C}, \quad \forall \{\sigma_\alpha\} \subseteq \Sigma_1,\ \{\sigma_\alpha\} \neq \emptyset. \tag{4}$$

Coercivity is a semi-continuity property that provides a smoothness constraint "from below" on the pyramid. The reverse implication in (4) actually holds for any pyramid of clusterings: from the connectivity-preserving property of clusterings and the pyramid property (3), it follows that, for $\psi_\tau(A) \in \mathcal{C}$ and $\sigma \geq \tau$, for some ρ we have $\psi_\sigma = \psi_\rho \psi_\tau$, so that $\psi_\sigma(A) = \psi_\rho(\psi_\tau(A)) \in \mathcal{C}$, which implies the desired result. Thus, it is the direct implication that is crucial for coercivity.

Consider the case when $\Sigma = [0, \infty]$ (so that $\Sigma_1 = [0, \infty)$). It can be verified that, in this case, coercivity is equivalent to

$$\psi_\tau(A) \in \mathcal{C},\ \text{for all } \tau > \sigma \;\Leftrightarrow\; \psi_\sigma(A) \in \mathcal{C}, \quad \text{for } \sigma \in [0, \infty). \tag{5}$$

In other words, A is a τ-cluster for every $\tau > \sigma$ if and only if A is a σ-cluster, for $\sigma \in [0, \infty)$. As before, the reverse implication is guaranteed for any pyramid of clusterings, regardless of coercivity. Note that any pyramid of clusterings is automatically coercive if the set of scales is discrete; e.g., if $\Sigma = \{0, 1, \ldots, K\}$.

The following proposition provides an alternative criterion for coercivity, in the case when the underlying connectivity class has $\downarrow$-*continuous* connectivity openings [6].

3.5 Proposition. Let $\mathcal{C}$ be a connectivity class in a lattice $\mathcal{L}$, with $\downarrow$-continuous connectivity openings. A pyramid of clusterings $\{\psi_\sigma \mid \sigma \in [0, \infty)\}$ on $\mathcal{L}$ is coercive if

$$\psi_\sigma = \bigwedge_{\tau > \sigma} \psi_\tau, \qquad \text{for } \sigma \in [0, \infty). \tag{6}$$

Proof. We need to show the direct implication in (5), since the reverse implication is guaranteed for a pyramid of clusterings. Given $\sigma \in [0, \infty)$ and $A \in \mathcal{L}$, suppose that $\psi_\tau(A) \in \mathcal{C}$, for all $\tau > \sigma$. If $\psi_\sigma(A) = O$, then $\psi_\sigma(A) \in \mathcal{C}$ and we are done. Otherwise, we may pick a sup-generator $x \leq \psi_\sigma(A)$. Now, it can be shown that if an operator ξ is $\downarrow$-continuous and $\{A_\sigma \mid \sigma \in \Sigma_1\}$ is an increasing family in $\mathcal{L}$, then $\xi(\bigwedge_{\tau > \sigma} A_\tau) = \bigwedge_{\tau > \sigma} \xi(A_\sigma)$, for $\sigma \in \Sigma_1$ (this is the dual of Proposition 2.2.10 in [5]). Since the connectivity opening γ_x is $\downarrow$-continuous and, as we argued previously, $\{\psi_\tau(A) \mid \tau > \sigma\}$ is an increasing family in $\mathcal{L}$, it follows that $\gamma_x \psi_\sigma(A) = \gamma_x(\bigwedge_{\tau > \sigma} \psi_\tau(A)) = \bigwedge_{\tau > \sigma} \gamma_x \psi_\tau(A) = \bigwedge_{\tau > \sigma} \psi_\tau(A) = \psi_\sigma(A)$, so that $\psi_\sigma(A) \in \mathcal{C}$, as required. Q.E.D.

In the spirit of second-generation connectivities, coercive pyramids of clusterings can be used to construct multiscale connectivities. Before we investigate how that can be done, we need to introduce the following definition. A *dual isomorphism* between two lattices $\mathcal{L}$ and $\mathcal{M}$ is a bijection $\nu\colon \mathcal{L} \to \mathcal{M}$ that reverses the partial ordering; i.e., $A \leq B$ if an only if $\nu(A) \geq \nu(B)$, for $A, B \in \mathcal{L}$. In this case, lattice $\mathcal{M}$ is referred to as the *negative* of lattice $\mathcal{L}$, and we write $\mathcal{M} = \mathcal{L}^*$. In addition, we write $\nu(A) = A^*$, for $A \in \mathcal{L}$. For example, we have $[0, \infty]^* = [-\infty, 0]$ and $\sigma^* = -\sigma$, for $\sigma \in [0, \infty]$. Moreover, the negative of the set $\{0, 1, \ldots, K\}$ is the same set but with reverse ordering, and $\sigma^* = K - \sigma$, for $\sigma = 0, 1, \ldots, K$. In what follows, we assume a scale lattice Σ and an associated negative scale lattice Σ^*. In addition, we write $\Sigma_0^* = \Sigma^* \setminus \{O_{\Sigma^*}\}$.

3.6 Proposition. Let $\mathcal{L}$ be a lattice, furnished with a connectivity class $\mathcal{C}$. If $\{\psi_\sigma \mid \sigma \in \Sigma_1\}$ is a coercive pyramid of clusterings on $\mathcal{L}$, then $\mathbf{C}\colon \Sigma_0^* \to \mathcal{P}(\mathcal{L})$, given by

$$\mathbf{C}(\sigma) = \psi_{\sigma^*}^{-1}(\mathcal{C}) = \{A \in \mathcal{L} \mid \psi_{\sigma^*}(A) \in \mathcal{C}\}, \quad \sigma \in \Sigma_0^*, \tag{7}$$

is a connectivity pyramid on $\mathcal{L}$, with associated connectivity measure φ: $\mathcal{L} \to \Sigma^*$ given by

$$\varphi(A) = \Big(\bigwedge\{\sigma \in \Sigma_1 \mid \psi_\sigma(A) \in \mathcal{C}\}\Big)^*, \quad A \in \mathcal{L}. \tag{8}$$

Proof. From the corresponding single-scale result [2, Proposition 6.5], it follows that $\mathbf{C}(\sigma)$ is a connectivity class in $\mathcal{L}$, for each $\sigma \in \Sigma_0^*$, which shows property (i) of a connectivity pyramid. To show property (ii), note that, for $\sigma \geq \tau$, $A \in \mathbf{C}(\sigma) = (\psi_{\sigma^*})^{-1}(\mathcal{C}) \Rightarrow \psi^{\sigma^*}(A) \in \mathcal{C}$ and, by the pyramid property, there exists a $\tau' \in \Sigma_0^*$ such that $\psi_{\tau^*}(A) = \psi_{\tau'}\psi_{\sigma^*}(A)$. But since $\psi_{\tau'}$ is connectivity-preserving, it follows that $\psi_{\tau^*}(A) \in \mathcal{C} \Rightarrow A \in \mathbf{C}(\tau)$, so that $\mathbf{C}(\sigma) \subseteq \mathbf{C}(\tau)$, as required. We now show property (iii). Given a nonempty set of scales $\{\sigma_\alpha\} \subseteq \Sigma_0^*$, we need to show that $A \in \mathbf{C}(\bigvee \sigma_\alpha) \Leftrightarrow A \in \mathbf{C}(\sigma_\alpha)$ for each σ_α. Using (7), this can be written as $\psi_{(\vee\sigma_\alpha)^*}(A) = \psi_{\wedge\sigma_\alpha^*}(A) \in \mathcal{C} \Leftrightarrow \psi_{\sigma_\alpha^*}(A) \in \mathcal{C}$, for each σ_α, which follows directly from the coercivity property. Finally, note that $\varphi(A) = \bigvee\{\sigma \in \Sigma_0^* \mid A \in \mathbf{C}(\sigma)\} = \bigvee\{\sigma \in \Sigma_0^* \mid \psi_{\sigma^*}(A) \in \mathcal{C}\} = (\bigwedge\{\sigma \in \Sigma_1 \mid \psi_\sigma(A) \in \mathcal{C}\})^*$, for $A \in \mathcal{L}$, which shows (8). Q.E.D.

In this framework, the degree of connectivity and the associated connectivity pyramid are "negative"; i.e., they are defined on the negative scale lattice Σ^*. The value of the degree of connectivity $\varphi(A)$ indicates how disconnected A is with respect to the base connectivity $\mathcal{C}$. The more "negative" (i.e., the smaller) the degree of connectivity is, the more disconnected A is, in the sense that a "larger" clustering needs to be applied on A in order to "reconnect" it.

As a direct consequence of the corresponding single-scale result [2, Proposition 6.6] and if the clusterings are strong, it follows that the σ-connectivity openings associated with $(\varphi, \mathbf{C})$ are given by

$$\gamma_x^\sigma(A) = \begin{cases} A \wedge \gamma_x\psi_{\sigma^*}(A), & \text{if } x \leq A \\ O, & \text{if } x \not\leq A \end{cases}, \quad A \in \mathcal{L},\ \sigma \in \Sigma_0^*,\ x \in \mathcal{S}, \tag{9}$$

where $\{\gamma_x \mid x \in \mathcal{S}\}$ are the connectivity openings associated with $\mathcal{C}$. If in addition $\mathcal{L}$ is infinite $\vee$-distributive, the σ-reconstruction operators associated with $(\varphi, \mathbf{C})$ are given by

$$\rho_\sigma(A \mid M) \ = \ A \wedge \rho(\psi_{\sigma^*}(A) \mid A \wedge M), \quad A, M \in \mathcal{L},\ \sigma \in \Sigma_0^*, \tag{10}$$

where ρ is the reconstruction operator associated with $\mathcal{C}$.

3.2 Examples

We now present several examples of multiscale connectivities based on pyramids of clusterings, which illustrate the previous theory.

3.2.1 Continuous-Scale Pyramid of Binary Dilations

As a first example, we provide a multiscale generalization of a single-scale connectivity class we have proposed in [2], generated by binary dilations (see also [9, 12]). Let E be a compact convex subset of $\mathbb{R}^n$, furnished with the Euclidean topology. Moreover, let $\mathcal{P}(E)$ be furnished with the connectivity class $\mathcal{C}$ of topologically connected sets in E, and let $B \subseteq \mathbb{R}^n$ be a closed Euclidean disk that contains the origin of $\mathbb{R}^n$. We define the *scaled replicas* of B by $\sigma B = \{\sigma b \mid b \in B\}$, for $\sigma \in [0, \infty)$. For a given $\sigma \in [0, \infty)$, consider the operator on $\mathcal{P}(E)$: $\delta_\sigma(A) = A \oplus_E \sigma B = (A \oplus \sigma B) \cap E$; i.e., the *conditional dilation* of A by σB inside E [6]. Using the facts that δ_σ is an extensive dilation, $\delta_\sigma(\{v\}) = \sigma B_v \cap E$ is connected for all $v \in E$, and $\mathcal{P}(E)$ is infinite $\vee$-distributive, we can apply Prop. 3.3 to conclude that δ_σ is a strong clustering, for the topologically connected sets in $\mathcal{P}(E)$. Moreover, Prop. 9.46 in [6] on metric geodesic dilations implies that $\delta_\sigma(A) = A \oplus_E \sigma B = (A \oplus_E \tau B) \oplus_E (\sigma - \tau)B = \delta_{\sigma-\tau}\delta_\tau(A)$, for $\sigma \geq \tau$, so that (3) is satisfied, and the family $\{\delta_\sigma(A) = A \oplus_E \sigma B \mid \sigma \in [0, \infty)\}$ is a pyramid of dilations on $\mathcal{P}(E)$.

The coercivity property cannot be established in $\mathcal{P}(E)$, and we need to focus our attention on the lattice $\mathcal{F}(E)$ of closed subsets of E instead. For the sake of conciseness, we will be relying on results shown in [5]. It follows from [6, Lemma 7.42] that, since A is closed and B is compact, $A \oplus_E B$ is closed. Therefore, by restriction, δ defines an operator $\hat{\delta}$ on $\mathcal{F}(E)$. However, $\hat{\delta}$ is a "pseudo-dilation" on $\mathcal{F}(E)$, because there instances in which it fails to commute with the supremum in $\mathcal{F}(E)$ (we will ignore this fine detail and refer to $\hat{\delta}$ as dilation). It is clear however that $\{\hat{\delta}_\sigma(A) = A \oplus_E \sigma B \mid \sigma \in [0, \infty)\}$ is a pyramid on $\mathcal{F}(E)$. The family $\hat{\mathcal{C}}$ of topologically connected closed sets in E, with the points as sup-generators, is a connectivity class in $\mathcal{F}(E)$ [5, Prop. 4.1.11]. From the facts that δ_σ is a strong clustering on $\mathcal{P}(E)$, that $\hat{\delta}_\sigma$ is the restriction of δ_σ to $\mathcal{F}(E)$, and that $\hat{\mathcal{C}} = \mathcal{C} \cap \mathcal{F}(E)$, it follows that $\hat{\delta}_\sigma$ is a strong clustering on $\mathcal{F}(E)$, for each $\sigma \in [0, \infty)$ [5, Lem. 6.3.7]. Therefore, $\{\hat{\delta}_\sigma(A) = A \oplus_E \sigma B \mid \sigma \in [0, \infty)\}$ is a pyramid of strong clusterings on $\mathcal{F}(E)$. To show the coercivity property, note that [5, Prop. 6.3.9]

$$A \oplus_E \sigma B = \bigcap_{\tau > \sigma} A \oplus_E \tau B, \quad A \in \mathcal{F}(E),\, \sigma \in [0, \infty)\,,$$

which implies $\hat{\delta}_\sigma = \bigwedge_{\tau > \sigma} \hat{\delta}_\tau$, for $\sigma \in [0, \infty)$, and that the connectivity openings associated with $\hat{\mathcal{C}}$ are $\downarrow$-continuous [5, Prop. 4.1.13]. The desired result then follows from Proposition 3.5 above.

The multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{F}(E)$, generated by this coercive pyramid of dilations, is such that $\mathbf{C}(\sigma) = \{A \in \mathcal{F}(E) \mid A \oplus_E |\sigma| B \text{ is topologically connected}\}$, for $\sigma \in (-\infty, 0]$, and $\varphi(A) = -\bigwedge\{\sigma \in [0, \infty) \mid A \oplus_E \sigma B \text{ is topologically connected}\}$, for $A \in \mathcal{F}(E)$. Hence, if A is

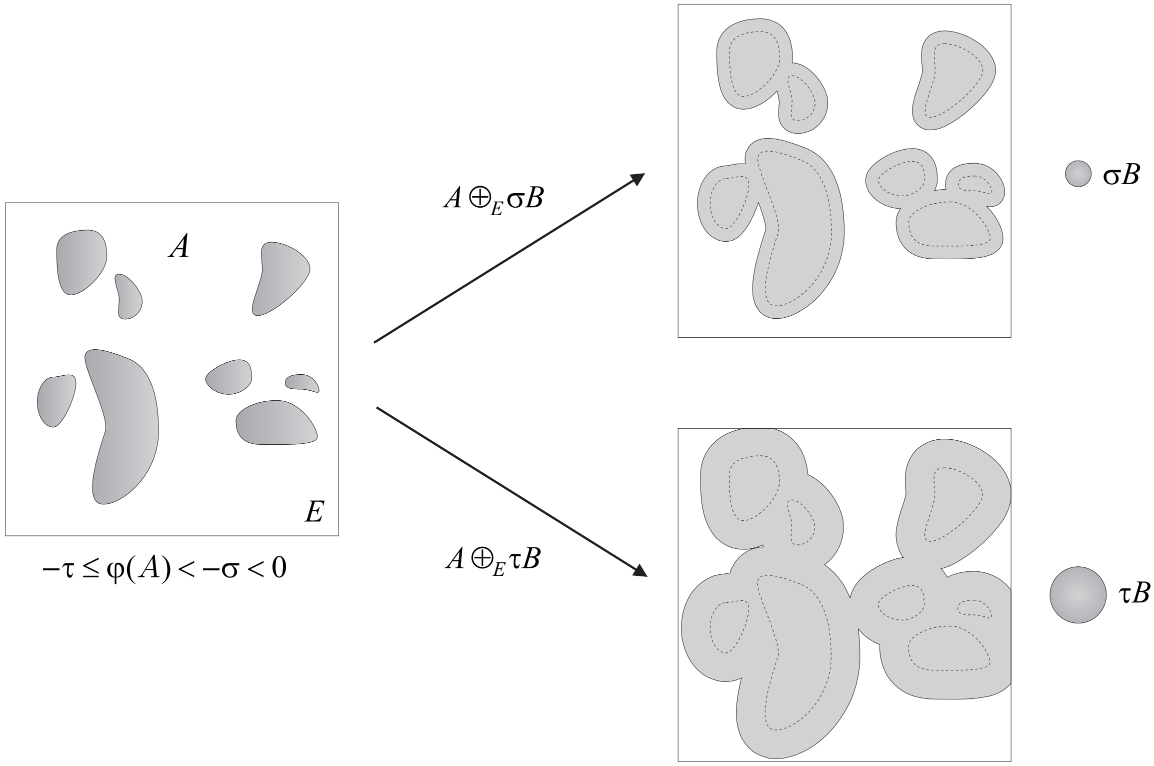

Figure 1: *Multiscale connectivity based on a continuous-scale pyramid of binary dilations. The original set A is topologically disconnected; hence its connectivity measure $\varphi(A)$ is negative. Note that $A \oplus_E \tau B$ is connected, but $A \oplus_E \sigma B$ is not. Therefore, A is $-\tau$-connected but not $-\sigma$-connected. Equivalently, $-\tau \leq \varphi(A) < -\sigma$ (see also Fig. 2 in [3]).*

topologically connected, then $\varphi(A)$ assumes its greatest value, $\varphi(A) = 0$, and A is fully connected. Otherwise, the degree of connectivity is strictly negative; it is the negative of the "size" of the "smallest" dilation required to dilate A so that it becomes topologically connected. See Fig. 1 for an illustration.

3.2.2 Discrete-Scale Pyramid of Binary Dilations

If the scale set is discrete, e.g., if $\Sigma = \{0, 1, \ldots, K\}$, then the coercivity requirement on the pyramid ceases to be an issue and the mathematical framework for developing multiscale connectivities becomes simpler.

Consider the lattice $\mathcal{L} = \mathcal{P}(\mathbb{Z}^n)$ (this example can be easily developed in $\mathcal{P}(\mathbb{R}^n)$ as well), with the points as sup-generators, furnished with an arbitrary translation-invariant connectivity class $\mathcal{C}$. Let $B \subseteq \mathbb{Z}^n$ be an arbitrary connected structuring element that contains the origin of $\mathbb{Z}^n$, and consider the family $\{\delta_\sigma(A) = A \oplus \sigma B \mid \sigma = 0, \ldots, K-1\}$ of dilations on $\mathcal{P}(\mathbb{Z}^n)$, where the scaled replicas of

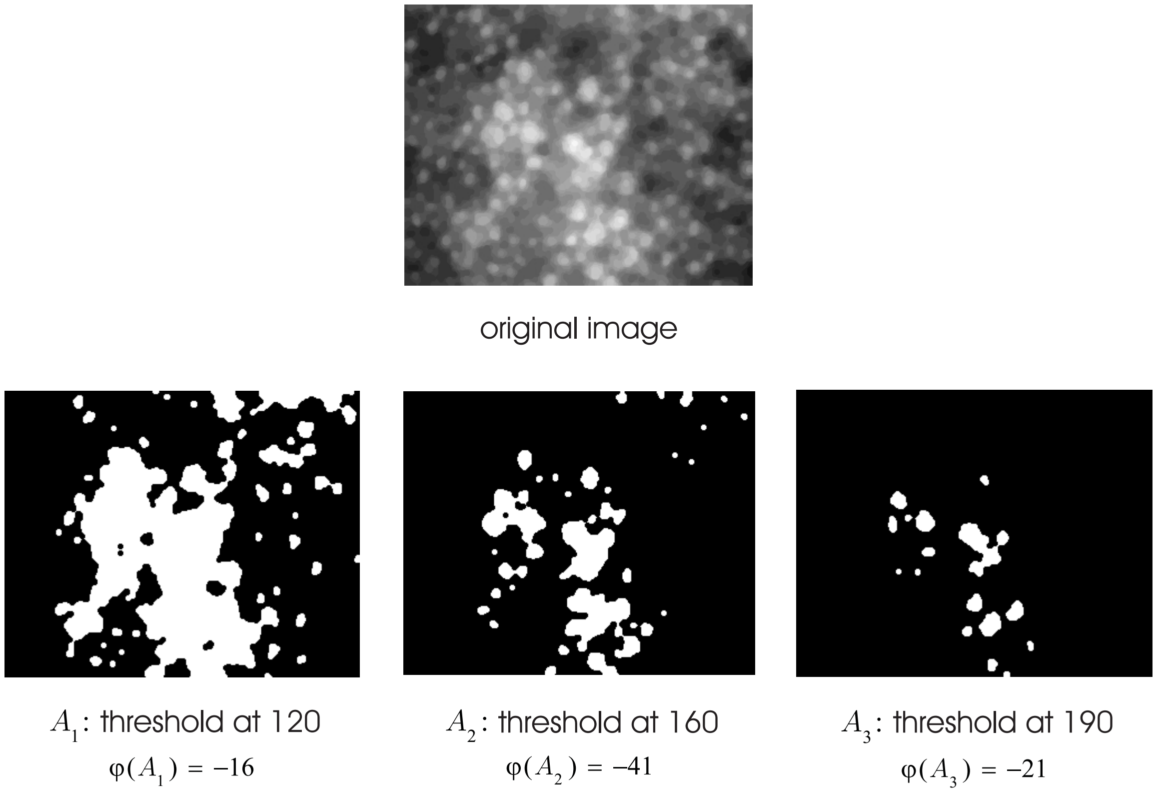

Figure 2: *Multiscale connectivity based on a discrete-scale pyramid of binary dilations. The original cornea cells image is thresholded with increasing threshold values to produce three binary images* A_1*,* A_2*, and* A_3*. Since the binary images are disconnected, their degrees of connectivity are negative. More negative values indicate a higher spread of individual image components.*

B are given by

$$\sigma B = \begin{cases} \underbrace{B \oplus B \oplus \cdots \oplus B}_{\sigma - 1 \text{ times}}, & \text{for } \sigma = 1, \ldots, K-1 \\ \{\mathbf{0}\}, & \text{for } \sigma = 0 \end{cases}, \tag{11}$$

with $\mathbf{0}$ denoting the origin of $\mathbb{Z}^n$. Using the facts that δ_σ is an extensive dilation, that $\delta_\sigma(\{v\}) = \sigma B_v \in \mathcal{C}$ for all $v \in E$, and that $\mathcal{P}(\mathbb{Z}^n)$ is infinite $\vee$-distributive, we can apply Prop. 3.3 to conclude that δ_σ is a strong clustering. Moreover, it follows from [6, Proposition 4.10] that $\delta_\sigma(A) = A \oplus \sigma B = (A \oplus \tau B) \oplus (\sigma - \tau)B = \delta_{\sigma-\tau}\delta_\tau(A)$, for $\sigma \geq \tau$, so that (3) is satisfied, and $\{\delta_\sigma(A) = A \oplus \sigma B \mid \sigma = 0, \ldots, K-1\}$ is a pyramid. The multiscale connectivity generated by this pyramid of dilations is illustrated in Fig. 2 with a simple example. We assume the usual 4-adjacency connectivity as the base connectivity and set B to be the 3×3 cross structuring element centered at the origin of $\mathbb{Z}^2$.

3.2.3 Pyramid of Grayscale Dilations

To develop this example, we need to briefly discuss an example of connectivity classes for grayscale images, which was introduced in [4].

We consider discrete grayscale images modeled as elements of the lattice $\mathcal{L} = \mathrm{Fun}(E, \mathcal{T})$, consisting of all functions from a domain of definition E into the discrete chain $\mathcal{T} = \{0, 1, \ldots, T\}$. Given an image $f \in \mathrm{Fun}(E, \mathcal{T})$ and a value $t \in \mathcal{T}$, the *level set* of f at level t is defined as the set $X_t(f) = \{v \in E \mid f(v) \geq t\}$. Given a connectivity class $\mathcal{C}$ in $\mathcal{P}(E)$, and for $t \in \mathcal{T}$, an image $f \in \mathrm{Fun}(E, \mathcal{T})$ is *level-t connected* if $X_s(f) \in \mathcal{C} \setminus \{\emptyset\}$, for all $s \leq t$. In other words, an image is level-t connected if all its level sets at or below level t are non-empty and connected. Loosely speaking, f is not allowed to have any "disconnecting dips" below level t.

A set $R \subseteq E$ is a *regional maximum* of $f \in \mathrm{Fun}(E, \mathcal{T})$ at level $t \in \mathcal{T}$ if R is a connected component of $X_t(f)$, according to $\mathcal{C}$, and $R \cap X_s(f) = \emptyset$, for all $s \geq t + 1$. We denote by $\mathcal{R}(f)$ the set of all regional maxima of an image f, and by $\mathcal{R}_t(f)$ the set of all regional maxima of f that are above level t; i.e., $\mathcal{R}_t(f) = \{R \in \mathcal{R}(f) \mid f(R) \geq t\}$, for $t \in \mathcal{T}$. Now, consider the subset of $\mathrm{Fun}(E, \mathcal{T})$ given by $\mathrm{Fun}_t(E, \mathcal{T}) = \{f \in \mathrm{Fun}(E, \mathcal{T}) \mid \mathcal{R}(f) = \mathcal{R}_t(f)\}$, for $t \in \mathcal{T}$. In other words, $\mathrm{Fun}_t(E, \mathcal{T})$ consists of the images that have all regional maxima above level t. It is shown in [4] that $\mathrm{Fun}_t(E, \mathcal{T})$ is an infinite $\vee$-distributive complete lattice, which is sup-generated by the family $\mathcal{S}_t$ consisting of all pulses of height at least t, along with the functions in $\mathrm{Fun}_t(E, \mathcal{T})$ that have exactly one regional maximum at level t. Moreover, it is shown that

$$\mathcal{C}_t = \{f \in \mathrm{Fun}(E, \mathcal{T}) \mid f \text{ is level-}t \text{ connected}\} \tag{12}$$

is a connectivity class in the lattice $\mathrm{Fun}_t(E, \mathcal{T})$, with sup-generating family $\mathcal{S}_t$.

Now, let $E = \mathbb{Z}^n$ and let $B \subseteq \mathbb{Z}^n$ be an arbitrary connected structuring element that contains the origin of $\mathbb{Z}^n$. Consider the family $\{\delta_\sigma(f) = f \oplus \sigma B \mid \sigma = 0, \ldots, K-1\}$ of *flat grayscale dilations* on $\mathrm{Fun}(E, \mathcal{T})$ [6], where the scaled replicas of $\mathcal{B}$ are given as in (11). It can be checked that this family constitutes a pyramid of strong clusterings in $\mathrm{Fun}_t(E, \mathcal{T})$, for a given $t \in \mathcal{T}$. It is easy to see that the multiscale connectivity $(\varphi, \mathbf{C})$ generated by this pyramid of dilations is such that

$$\mathbf{C}(\sigma) = \{f \in \mathrm{Fun}_t(E, \mathcal{T}) \mid f \text{ is level-}t \text{ connected according to } \widetilde{\mathcal{C}}_\sigma\},$$

for $\sigma = -(K-1), \ldots, -1, 0$, where $\widetilde{\mathcal{C}}_\sigma$ is the second-generation connectivity class in $\mathcal{P}(E)$ induced by the binary dilation $A \oplus |\sigma| B$, with base connectivity $\mathcal{C}$.

The grayscale σ-connected components of an image $f \in \mathrm{Fun}_t(E, \mathcal{T})$ according to $(\varphi, \mathbf{C})$, corresponding to a given marker in $\mathcal{S}_t$, can be computed in terms of the connectivity openings for $\mathcal{C}_t$ using

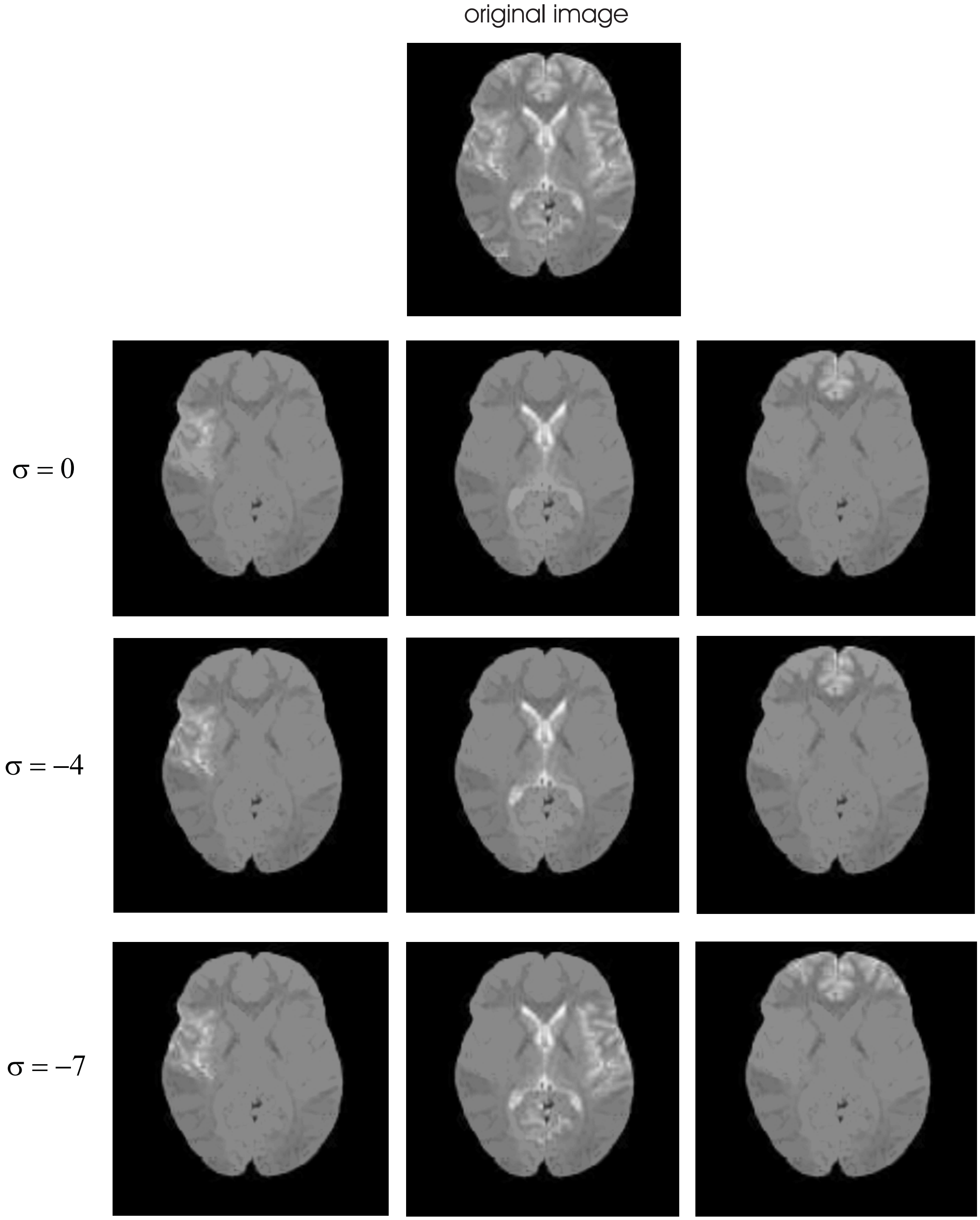

Figure 3: *Multiscale connectivity based on a discrete-scale pyramid of grayscale dilations. A few σ-connected components of the original image are shown for different values of the connectivity scale σ.*

(9), due to the fact that the clusterings are strong (refer to [4] for a discussion on how to compute connectivity components for $\mathcal{C}_t$). Figure 3 illustrates this case on a brain MR image, which has been pre-filtered to remove all regional maxima below level t and bring it into $\mathrm{Fun}_t(E, \mathcal{T})$. Here, $t = 172$, the base connectivity $\mathcal{C}$ corresponds to the usual 4-adjacency connectivity, and B is the 3×3 cross structuring element centered at the origin of $\mathbb{Z}^2$. A few grayscale σ-connected components of the original image are displayed, for different values of the connectivity scale parameter. The top row corresponds to the original grayscale level connectivity ($\sigma = 0$). Each column depicts three connected components at the corresponding scale. At smaller scales, the connected components are larger, since many adjacent level components are grouped together. As the scale decreases, the connected components in each column become increasingly larger by progressively including neighboring structures, as determined by the underlying dilation $\delta_\sigma(f) = f \oplus \sigma B$. This simple example shows that the multiscale connectivity based on pyramids of grayscale dilations may be used to decompose an image into clusters of neighboring structures, by progressively increasing the size of the clusters as the scale decreases. In the case depicted in Fig. 3, the connectivity scale can select for larger or smaller brain structures in the image.

3.2.4 Anti-Granulometries

Dilations are not the only operators that produce clusterings or pyramids. It is well-known that increasing families of closings, known as *anti-granulometries* [6], form pyramids [16] as well. In addition, connectivity-preserving closings are known to be clusterings [2, 14].

To be more specific, let $\mathcal{L}$ be an arbitrary lattice, furnished with an arbitrary connectivity class $\mathcal{C}$. Consider a discrete scale lattice, e.g., $\Sigma = 0, 1, \ldots, K$, and an anti-granulometry $\{\phi_\sigma \mid \sigma = 0, \ldots, K-1\}$ on $\mathcal{L}$, i.e., a family such that $\phi_\tau \geq \phi_\sigma$, for $\tau \geq \sigma$. It follows from [6, Theorem 3.24] that $\phi_\tau = \phi_\tau \phi_\sigma$, for $\tau \geq \sigma$, so that (3) is satisfied, and $\{\phi_\sigma \mid \sigma \in \sigma = 0, \ldots, K-1\}$ is a pyramid. If, in addition, each closing ϕ_σ is connectivity preserving, i.e., if $\psi_\sigma(\mathcal{C}) \subseteq \mathcal{C}$, for $\sigma = 0, \ldots, K-1$, then ϕ_σ is a clustering by [2, Proposition 6.8] (not necessarily a strong one), and thus we obtain a pyramid of clusterings. Structural closings by linear structuring elements [2] and by combinations of linear structuring elements at varying directions are examples of connectivity-preserving closings [15]. Multiscale connectivities can thus be constructed from such anti-granulometries.

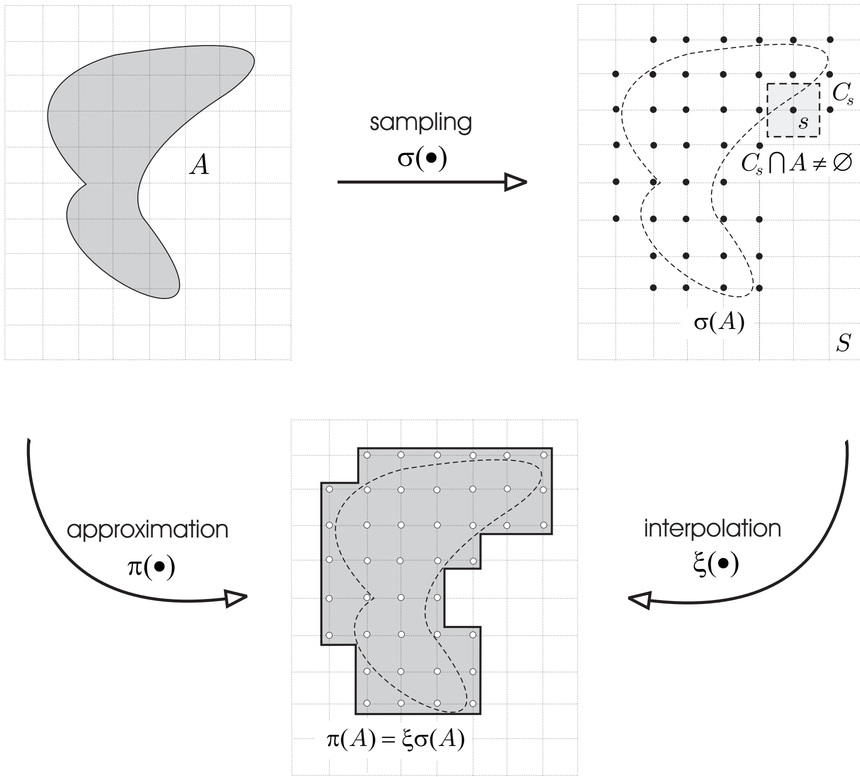

Figure 4: *Morphological sampling and interpolation (from [2]).*

3.2.5 Pyramid of Sampling Operators

We now describe a specific example of multiscale connectivity based on anti-granulometry, which is based on the theory of *morphological sampling* (this example was originally suggested to us by H. Heijmans – see also [2]). In the following, we briefly review basic aspects of this theory. For a general treatment, the reader is referred to [6, 7, 17].

Consider the lattice $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$, with the points as sup-generators. Let $S \subset \mathbb{R}^n$ be a regular grid in $\mathbb{R}^n$, which is known as the *sampling grid*. Let $C \subset \mathbb{R}^n$ be a bounded set such that $\mathbf{0} \in C$, $C \cap S = \{\mathbf{0}\}$ and $S \oplus C = \mathbb{R}^n$, where $\mathbf{0}$ is the origin in $\mathbb{R}^n$. The set C is known as the *sampling element*. One can show that the operator $\sigma(A) = \{s \in S \mid C_s \cap A \neq \emptyset\}$, known as the *sampling operator*, defines a dilation from $\mathcal{P}(\mathbb{R}^n)$ into $\mathcal{P}(S)$. The corresponding adjoint erosion from $\mathcal{P}(S)$ into $\mathcal{P}(\mathbb{R}^n)$, called the *interpolation operator*, is given by $\xi(V) = \left(\bigcup_{s \in S \setminus V} C_s\right)^c$. The sampling operator followed by the interpolation operator produces an operator on $\mathcal{P}(\mathbb{R}^n)$, given by $\pi(A) = \xi\sigma(A) = \left(\bigcup_{s \in S}\{C_s \mid C_s \cap A = \emptyset\}\right)^c$, which is called the *approximation closing*. Figure 4 illustrates this morphological sampling scheme.

Consider now two families $\{S_\sigma \mid \sigma = 0, 1, \ldots, K-1\}$ and $\{C_\sigma \mid \sigma = 0, 1, \ldots, K-1\}$ of increasingly coarser sampling grids and sampling elements, respectively, given by $S_\sigma = 2^\sigma S$ and $C_\sigma = 2^\sigma C$, for $\sigma = 0, 1, \ldots, K-1$, where $S = \{k_1 u_1 + \cdots + k_n u_n \mid k_i \in \mathbb{Z}\}$ and $C = \{x_1 u_1 + \cdots + x_n u_n \mid -1 < x_i < 1\}$, with $\{u_i \mid i = 1, 2, \ldots, n\}$ being linearly independent vectors in $\mathbb{R}^n$ (for example, the case when $\{u_i \mid i = 1, 2, \ldots, n\}$ is the orthonormal basis of $\mathbb{R}^n$ corresponds to the most commonly used sampling scheme). Let $\{\pi_\sigma \mid \sigma = 0, 1, \ldots, K-1\}$ be the associated family of approximation closings, given by

$$\pi_\sigma(A) = \left(\bigcup_{s \in S_\sigma} \{(C_\sigma)_s \mid (C_\sigma)_s \cap A = \emptyset\} \right)^c, \quad A \in \mathcal{P}(\mathbb{R}^n). \tag{13}$$

Given a set $A \in \mathcal{P}(\mathbb{R}^n)$, $\pi_\sigma(A)$ provides an approximation of A at scale σ. Note that $S_0 = S$ and $C_0 = C$. The corresponding approximation closing $\pi_0(A)$, which is referred to as the *basic discretization* of A, provides the finest available discretization of A. We have the following result.

3.7 Proposition. The family $\{\pi_\sigma \mid \sigma = 0, 1, \ldots, K-1\}$ of approximation closings is an anti-granulometry on $\mathcal{P}(\mathbb{R}^n)$.

PROOF. We need to show that $\pi_\sigma \geq \pi_\tau$, for $\sigma \geq \tau$. Given $A \in \mathcal{L}$, we show that $(\pi_\sigma(A))^c \subseteq (\pi_\tau(A))^c$. For clarity, we use the notation $C_\sigma(s) = (C_\sigma)_s$. Let $v \in (\pi_\sigma(A))^c$. From (13), it follows that this is equivalent to the fact that there is an $s_0 \in S_\sigma$ such that $v \in C_\sigma$, with $C_\sigma \cap A = \emptyset$. Now, it is easy to verify that $C_\sigma(s) = \bigcup_{s' \in S_\tau} \{C_\tau(s') \mid C_\tau(s') \subseteq C_\sigma(s)\}$, for any $s \in S_\sigma$; i.e., the larger sampling element $C_\sigma(s)$ equals a union of appropriately translated smaller sampling elements $C_\tau(s')$. Therefore, we can find an $s_1 \in S_\tau$ such that $v \in C_\tau(s_1) \subseteq C_\sigma(s_0)$, so that $v \in C_\tau(s_1) \cap A = \emptyset$. In other words, $v \in (\pi_\tau(A))^c$, as required. Q.E.D.

We define a connectivity class $\mathcal{C}$ in $\mathcal{P}(\mathbb{R}^n)$ to be *scaling-invariant* if $A \in \mathcal{C} \Leftrightarrow \sigma A \in \mathcal{C}$, for any positive scale parameter σ. For instance, the Euclidean topological connectivity class in $\mathcal{P}(\mathbb{R}^n)$ is scaling-invariant. It was shown in [2, Proposition 6.11] that, if $C_s \setminus (R \oplus C) \in \mathcal{C}$, for all $s \in S$ and $R \subseteq S$, the approximation closing $\pi(A) = \left(\bigcup_{s \in S} \{C_s \mid C_s \cap A = \emptyset\}\right)^c$ is connectivity-preserving, and hence a clustering. The required condition is easy to check in practice [2]. This result and the scale invariance of $\mathcal{C}$ directly lead to following proposition.

3.8 Proposition. Consider the lattice $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$, furnished with a scaling-invariant connectivity class $\mathcal{C}$, such that

$$C_s \setminus (R \oplus C) \in \mathcal{C}, \quad \forall s \in S,\ \forall R \subseteq S. \tag{14}$$

Then, each approximation closing π_σ is connectivity-preserving, and thus a clustering, for $\sigma = 0, 1, \ldots, K-1$. Hence, the anti-granulometry $\{\pi_\sigma \mid \sigma = 0, 1, \ldots, K-1\}$ is a pyramid of clusterings on $\mathcal{P}(\mathbb{R}^n)$.

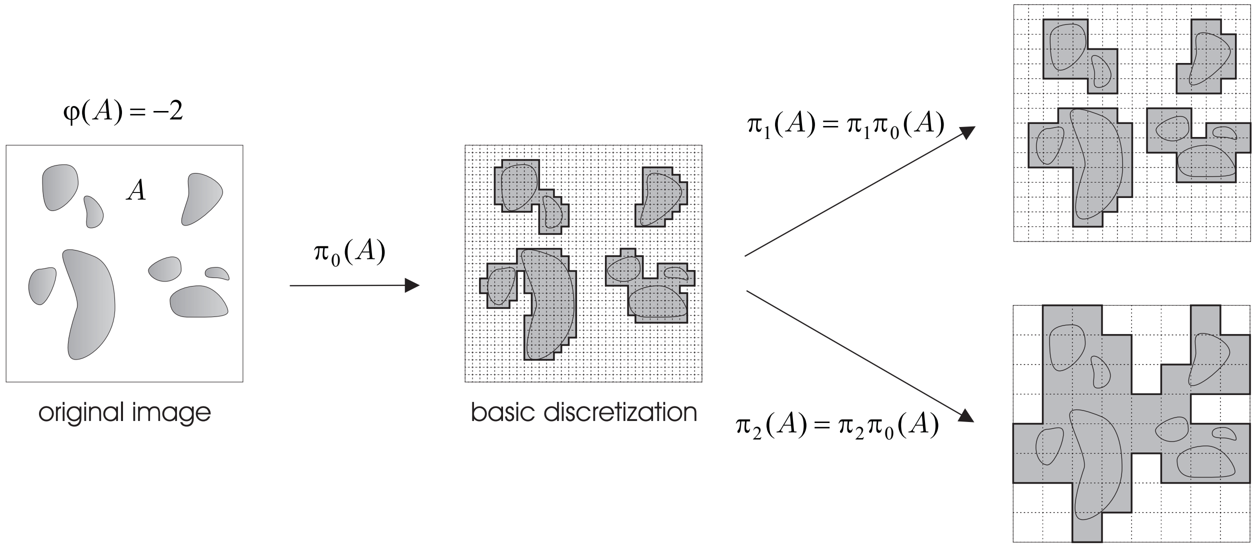

Figure 5: *Multiscale connectivity based on a pyramid of sampling operators. The original image A and its basic discretization $\pi_0(A)$ are disconnected; hence, its degree of connectivity $\varphi(A)$ is strictly negative. Note that $\pi_2(A)$ is connected, but $\pi_1(A)$ is not. Therefore, A is -2-connected but not -1-connected. It follows that $\varphi(A) = -2$.*

The multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{P}(\mathbb{R}^n)$, generated by this pyramid of sampling operators, is such that $\mathbf{C}(\sigma) = \{A \subseteq \mathbb{R}^n \mid \pi_\sigma(A) \in \mathcal{C}\}$, for $\sigma = -(K-1), \ldots, -1, 0$, and $\varphi(A) = -\min\{\sigma = 0, 1, \ldots, K \mid \pi_\sigma(A) \in \mathcal{C}\}$, for $A \subseteq \mathbb{R}^n$. Hence, $\varphi(A) = 0$, if either A or its basic discretization $\pi_0(A)$ is connected in $\mathcal{C}$, in which case A is fully connected. Otherwise, the degree of connectivity is strictly negative: it is the negative of the minimum discretization scale at which A is connected. Figure 5 illustrates this multiscale connectivity. Note that increasingly coarser discretizations of A eventually produce a connected image. The (negative) degree of connectivity of A measures how disconnected A is, with respect to the base connectivity $\mathcal{C}$.

4 Multiscale Connectivities Based on Granulometries

In this section, we present another class of multiscale connectivities, based on granulometries. These are "positive" multiscale connectivities, in contrast to the "negative" multiscale connectivities generated by clustering pyramids. Here, if an element is not connected with respect to the base connectivity, then its degree of connectivity is zero and that element is fully disconnected. Otherwise, the degree of connectivity is positive, and the more positive it is, the "more connected" the element is with respect to the base connectivity.

4.1 Basic Notions

Given a base connectivity class $\mathcal{C}$ and an opening θ, second-generation connectivity classes in $\mathcal{P}(E)$ were constructed in [8, 9] as union of the empty set, the singletons, and $\mathcal{C} \cap \text{Inv}(\theta)$. This framework was later generalized to arbitrary atomic lattices in [2, 5]. Below, we show how this can be exploited for constructing multiscale connectivities.

Similarly to Section 3, we first provide basic definitions and results concerning multiscale connectivities based on granulometries and then consider two examples. A *granulometry* is simply a family of openings $\{\theta_\sigma \mid \sigma \in J\}$ on a lattice $\mathcal{L}$, where J is an indexing poset, with the property $\theta_\tau \leq \theta_\sigma$, for $\tau \geq \sigma$ [6]. Granulometries are pyramids of openings; it follows from [6, Theorem 3.24] that $\theta_\tau = \theta_\tau\theta_\sigma$, for $\tau \geq \sigma$, so that (3) is satisfied.

Given a granulometry $\{\theta_\sigma \mid \sigma \in \Sigma_0\}$ on $\mathcal{L}$, we say that $A \in \mathcal{L}$ is σ*-open* if $\xi_\sigma(A) = A$, for $\sigma \in \Sigma_0$. Moreover, a granulometry is said to be *locally invariant* with respect to a connectivity class $\mathcal{C}$ in $\mathcal{L}$ if each opening is locally invariant with respect to $\mathcal{C}$; i.e., if $\theta_\sigma(A) = A \;\Rightarrow\; \theta_\sigma\gamma_x(A) = \gamma_x(A)$, for $\sigma \in \Sigma_0$, $x \in \mathcal{S}$ and $A \in \mathcal{L}$. In other words, a granulometry on $\mathcal{L}$ is locally invariant if, for a given σ-open $A \in \mathcal{L}$, each connected component of A is also σ-open, for $\sigma \in \Sigma_0$. We will see below that, similarly to the strong property of clusterings,the local invariance property for openings is useful for characterizing σ-connectivity openings and σ-reconstruction operators associated with the generated multiscale connectivity. We remark that the property of local invariance of openings was first introduced in [8], where it was referred to by saying that "the opening is connected for $\mathcal{C}$." However, this terminology eventually became undesirable with the subsequent development of the theory of *connected operators* [10].

The coercivity property (see Defn. 3.4) has the following counterpart for granulometries.

4.1 Definition. A granulometry $\{\theta_\sigma \mid \sigma \in \Sigma_0\}$ is said to be *coercive* if:

$$\text{Inv}(\theta_{\vee\sigma_\alpha}) = \bigcap \text{Inv}(\theta_{\sigma_\alpha}), \quad \forall\{\sigma_\alpha\} \subseteq \Sigma_0,\ \{\sigma_\alpha\} \neq \emptyset. \tag{15}$$

We use $\Sigma_0 = \Sigma \setminus \{O_\Sigma\}$ instead of Σ since it will become clear by Prop. 4.3 that the bottom scale O_Σ is not needed for the construction of the multiscale connectivities based on granulometries.

Coercivity is a semi-continuity property that provides a smoothness constraint "from above" on the granulometry. By using the anti-extensivity of openings, it can be shown easily that the direct inclusion in (15) holds. Hence, it is the reverse inclusion that is crucial for coercivity of a granulometry.

Consider the case when $\Sigma = [0,\infty]$ (so that $\Sigma_0 = (0,\infty]$). In this case, coercivity of a granulometry can be verified to be equivalent to

$$\mathrm{Inv}(\theta_\sigma) = \bigcap_{\tau<\sigma} \mathrm{Inv}(\theta_\tau), \qquad \text{for } \sigma \in (0,\infty]. \tag{16}$$

In other words, A is σ-open if and only if A τ-open for every $\tau < \sigma \in (0,\infty]$. Note that the direct implication is true for any granulometry, regardless of coercivity. We remark that any granulometry is automatically coercive if the set of scales is discrete; e.g., if $\Sigma = \{0,1,\ldots,K\}$. The following proposition, which is the dual of Proposition 3.5, provides an alternative criterion for coercivity of a granulometry.

4.2 Proposition. A granulometry $\{\theta_\sigma \mid \sigma \in \Sigma_0\}$ is coercive if

$$\theta_{\vee\sigma_\alpha} = \bigwedge \theta_{\sigma_\alpha}, \quad \forall\{\sigma_\alpha\} \subseteq \Sigma_0,\ \{\sigma_\alpha\} \neq \emptyset. \tag{17}$$

Proof. Since all operators involved are anti-extensive, we have that $\theta_\sigma(A) = \bigwedge \theta_{\sigma_\alpha}(A) = A$ if and only if $\theta_{\sigma_\alpha}(A) = A$, for all σ_α, from which (15) follows. Q.E.D.

Note that when $\Sigma = [0,\infty]$, then (17) reduces to

$$\theta_\sigma = \bigwedge_{\tau<\sigma} \theta_\tau, \quad \sigma \in (0,\infty]. \tag{18}$$

We remark that (15) does not imply (17), i.e., a coercive granulometry does not need to satisfy this property. In fact, the infimum of openings is not even necessarily an opening. What can be shown (see [5, Prop. 6.3.21]) is that (15) is equivalent to the weaker condition $\theta_{\vee\sigma_\alpha} = (\bigwedge \theta_{\sigma_\alpha})^\circ$, for any nonempty set of scales $\{\sigma_\alpha\} \subseteq \Sigma_0$, where $\psi^\circ(A) = \bigvee\{B \in \mathrm{Inv}(\psi) \mid B \leq A\}$ defines the *characteristic opening* associated with an operator ψ [5]. It can be shown that $\psi^\circ = \psi$ if and only if ψ is an opening itself [5, Prop. 2.2.6]. Therefore, if for a coercive granulometry, the infimum of any subfamily of its openings is an opening, then (17) is necessarily satisfied. For example, it can be shown [3, Thm. 4.1] that, for each fixed $x \in \mathcal{S}$, the family of σ-connectivity openings $\{\gamma_x^\sigma \mid \sigma \in \Sigma_0\}$ associated with an arbitrary multiscale connectivity is a granulometry that satisfies the property $\gamma_x^{\vee\sigma_\alpha} = (\bigwedge \gamma_x^{\sigma_\alpha})^\circ$, for any nonempty set of scales $\{\sigma_\alpha\} \subseteq \Sigma_0$; hence, $\{\gamma_x^\sigma \mid \sigma \in \Sigma_0\}$ is a coercive granulometry.

Coercive granulometries can be used to construct multiscale connectivities on atomic lattices. This is shown by the next result (see also [9]).

4.3 Proposition. Let $\mathcal{L}$ be an atomic lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. If $\{\theta_\sigma \mid \sigma \in \Sigma_0\}$ is a coercive granulometry on $\mathcal{L}$, then $\mathbf{C}\colon \Sigma_0 \to \mathcal{P}(\mathcal{L})$, given by:

$$\mathbf{C}(\sigma) = \{O\} \cup \mathcal{S} \cup \{A \in \mathcal{C} \mid \theta_\sigma(A) = A\}, \quad \sigma \in \Sigma_0 \tag{19}$$

is a connectivity pyramid on $\mathcal{L}$, with associated connectivity measure φ: $\mathcal{L} \to \Sigma$ given by:

$$\varphi(A) = \begin{cases} I_\Sigma, & \text{if } A = O \text{ or } A \in \mathcal{S} \\ \bigvee\{\sigma \in \Sigma_0 \mid \theta_\sigma(A) = A\}, & \text{if } A \in \mathcal{C} \setminus (\{O\} \cup \mathcal{S}) \\ O_\Sigma, & \text{if } A \notin \mathcal{C} \end{cases}, \quad A \in \mathcal{L}. \tag{20}$$

Proof. From the single-scale version of this result [2, Proposition 7.1], it follows that, for each $\sigma \in \Sigma$, $\mathbf{C}(\sigma)$ is a connectivity class in $\mathcal{L}$, which shows property (i) of a connectivity pyramid. From the fact that θ_σ and θ_τ are anti-extensive operators and for $\sigma \geq \tau$, we have that $\theta_\sigma \leq \theta_\tau \Rightarrow \text{Inv}(\theta_\sigma) \subseteq \text{Inv}(\theta_\tau) \Rightarrow \mathbf{C}(\sigma) = \{O\} \cup \mathcal{S} \cup (\mathcal{C} \cap \text{Inv}(\theta_\sigma)) \subseteq \{O\} \cup \mathcal{S} \cup (\mathcal{C} \cap \text{Inv}(\theta_\tau)) = \mathbf{C}(\tau)$, which shows property (ii). We now show property (iii). Given a nonempty set of scales $\{\sigma_\alpha\} \subseteq \Sigma_0$, we need to show that $A \in \mathbf{C}(\bigvee \sigma_\alpha) \;\Leftrightarrow\; A \in \mathbf{C}(\sigma_\alpha)$, for each σ_α. But this follows directly from (19) and the coercivity property (15). Finally, (20) follows easily from (2) and (19). Q.E.D.

In contrast to multiscale connectivities based on pyramids of clusterings, we use here the original scale lattice instead of its negative. For $A \in \mathcal{C} \setminus (\{O\} \cup \mathcal{S})$, the degree of connectivity indicates how stable A is with respect to the granulometry. In this case, the more "positive" (i.e., the larger) the degree of connectivity is, the more connected A is, in the sense that a "larger" opening needs to be applied on A in order to disconnect it.

As a straightforward generalization of the corresponding single-scale result [2, Proposition 7.5], if the granulometry is locally invariant, then σ-connectivity openings associated with $(\varphi, \mathbf{C})$ are given by

$$\gamma_x^\sigma(A) = \begin{cases} \gamma_x\theta_\sigma(A), & \text{if } \; x \leq \theta_\sigma(A) \\ x, & \text{if } \; \theta_\sigma(A) \not\geq x \leq A \\ O, & \text{if } \; x \not\leq A \end{cases}, \quad A \in \mathcal{L},\; \sigma \in \Sigma_0,\; x \in \mathcal{S}.$$

where $\{\gamma_x \mid x \in \mathcal{S}\}$ are the connectivity openings associated with $\mathcal{C}$. If in addition $\mathcal{L}$ is infinite $\vee$-distributive, the σ-reconstruction operators associated with $(\varphi, \mathbf{C})$ are given by

$$\rho_\sigma(A \mid M) = (A \wedge M) \vee \rho(\theta_\sigma(A) \mid M), \quad A, M \in \mathcal{L},\; \sigma \in \Sigma_0.$$

where ρ is the reconstruction operator associated with $\mathcal{C}$.

Before, we proceed, we have an important observation. A *contraction* is any increasing and anti-extensive operator (this terminology appears in [8]). Similarly to what was done in [2, 5], most of the development presented in this section goes through unchanged if one replaces granulometries (i.e., pyramids of openings) by the more general concept of pyramids of contractions (thus, the idempotence property of openings is not essential here).

4.2 Examples

In this subsection, we give two examples of multiscale connectivities based on granulometries, which illustrate the theory presented above.

4.2.1 Continuous-Scale Granulometry

Let E be a compact convex subset of $\mathbb{R}^n$, furnished with the Euclidean topology. Moreover, let $\mathcal{L} = \mathcal{F}(E)$, with the points as sup-generators, furnished with the connectivity class $\mathcal{C}$ of topologically connected closed sets in E, and let $B \subseteq \mathbb{R}^n$ be a closed Euclidean disk. Consider the family of openings $\{\theta_\sigma(A) = A \circ \sigma B \mid \sigma \in (0, \infty]\}$ on $\mathcal{P}(E)$. Since, for $\tau \geq \sigma$, we have $\tau B = (\tau - \sigma)B \oplus \sigma B$, it follows that τB is σB-open, and $A \circ \tau B \subseteq A \circ \mathcal{B}$ [6, Props. 4.21-4.22]. Hence, $\{\theta_\sigma(A) \mid \sigma \in (0, \infty]\}$ is a granulometry on $\mathcal{P}(E)$. Moreover, since B is closed and E is bounded, B is compact in $\mathbb{R}^n$, and so is σB, for $\sigma \in (0, \infty]$. However, since A is closed in $\mathbb{R}^n$, it follows from [6, Lemma 7.42] that $A \circ \sigma B$ is closed in $\mathbb{R}^n$, so that $A \circ \sigma B \in \mathcal{F}(E)$, for $\sigma \in (0, \infty]$. Therefore, the restriction of the openings $\{\theta_\sigma(A) \mid \sigma \in (0, \infty]\}$ to $\mathcal{F}(E)$ is a granulometry on $\mathcal{F}(E)$. Coercivity of this granulometry can be shown in the following way. It can be shown [5, Prop. 6.3.22] that

$$A \circ \sigma B = \bigcap_{\tau < \sigma} A \circ \tau B, \quad A \in \mathcal{F}(E),\, \sigma \in (0, \infty]\,,$$

so $\theta_\sigma = \bigwedge_{\tau < \sigma} \theta_\tau$, for $\sigma \in (0, \infty]$. The desired result follows then from Prop. 4.2 and (18). Moreover, it can be shown that this granulometry is locally invariant with respect to $\mathcal{C}$ [5, Prop. 6.3.25]. This example generalizes an idea in [9].

The multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{F}(E)$ generated by this coercive granulometry is such that

$$\mathbf{C}(\sigma) = \{\emptyset\} \cup \mathcal{S} \cup \{A \in \mathcal{F}(E) \mid A \circ \sigma B = A\}, \quad \text{for } \sigma \in (0, \infty],$$

and

$$\varphi(A) = \begin{cases} \infty, & \text{if } A = \emptyset \text{ or } A \text{ is a point} \\ \bigvee\{\sigma \in (0, \infty] \mid A \circ \sigma B = A\}, & \text{if } A \in \mathcal{C} \setminus (\{O\} \cup \mathcal{S}) \\ 0, & \text{if } A \notin \mathcal{C} \end{cases}, \quad A \in \mathcal{L}.$$

Hence, the degree of connectivity is the "size" of the "largest" opening required to remove thin parts of A with respect to the structuring element used. These thin parts can be interpreted as "necks" or "bridges" in A, whose deletion disconnects A. See Fig. 6 for an illustration.

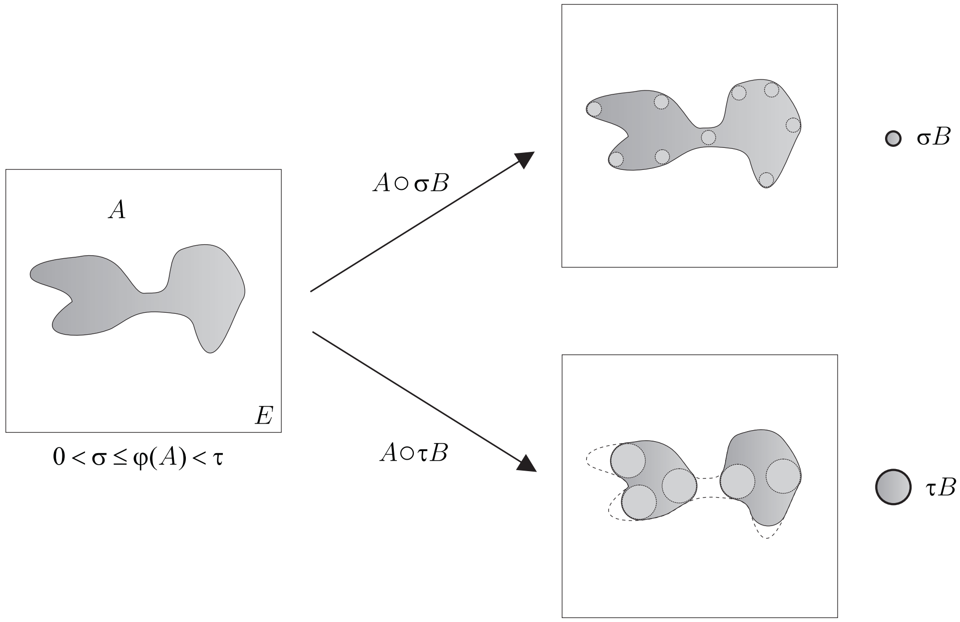

Figure 6: *Multiscale connectivity based on a continuous-scale granulometry. The original set A is topologically connected, and it is neither empty nor a point; hence, its degree of connectivity $\varphi(A)$ is nonzero and finite. Note that $A \circ \sigma B = A$, but $A \circ \tau B$ removes thin parts of A, including the "bridge" in A. Therefore, A is σ-connected, but not τ-connected. Equivalently, $\sigma \leq \varphi(A) < \tau$ (see also Fig. 3 in [3].)*

4.2.2 Discrete-Scale Granulometry

As in the case of pyramids of clusterings, the mathematical framework is simplified substantially if the scale set is discrete, e.g., when $\Sigma = \{0, 1, \ldots, K\}$. In this case, the coercivity requirement on the granulometries ceases to be an issue.

Let $\mathcal{L} = \mathcal{P}(\mathbb{Z}^n)$ (this example can be easily developed in $\mathcal{P}(\mathbb{R}^n)$ as well), with the points as sup-generators, furnished with an arbitrary translation-invariant connectivity class $\mathcal{C}$. If B is a connected structuring element, then the family $\{\theta_\sigma(A) = A \circ \sigma B \mid \sigma = 1, 2, \ldots, K\}$, where the replicas $\mathcal{B}$ are defined as in (11), is clearly a granulometry on $\mathcal{P}(\mathbb{Z}^n)$. Moreover, it can be shown to be locally invariant [5, Cor. 4.3.19].

Figure 7 illustrates this example, where we consider a discrete images and assume the usual 4-adjacency connectivity. The structuring element B is taken to be a horizontal line segment of length 2.

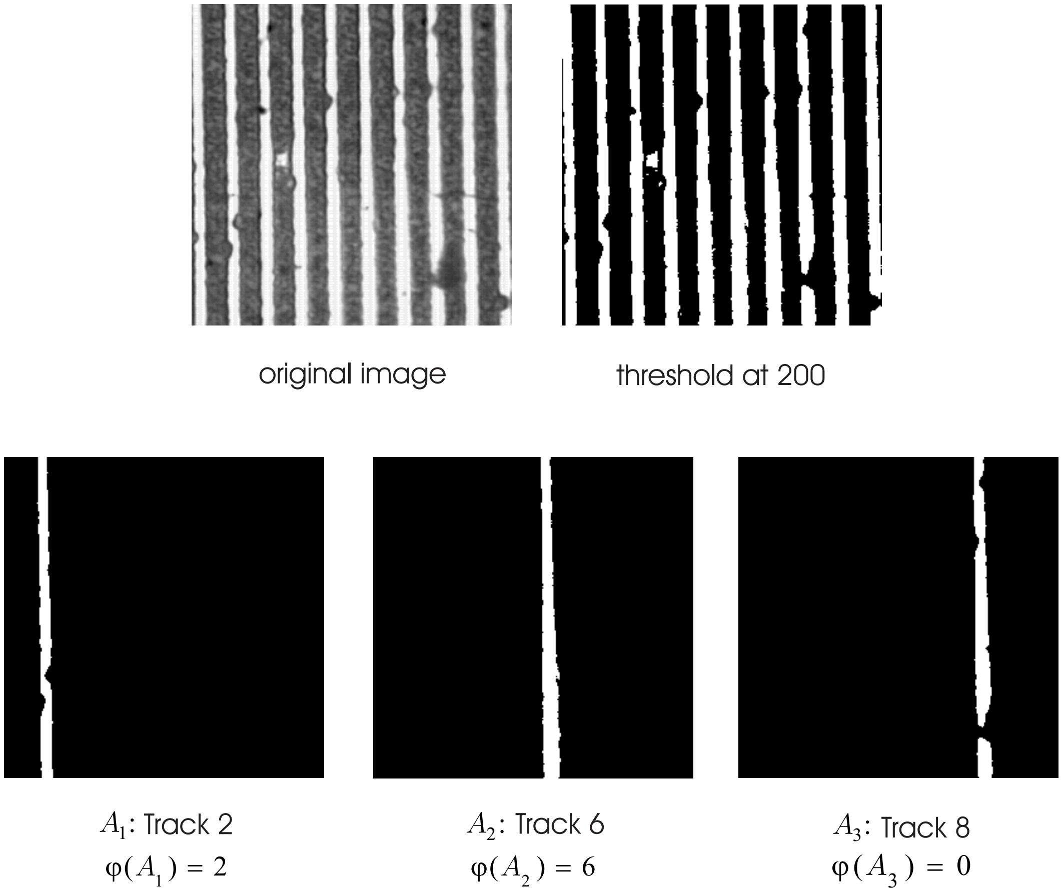

Figure 7: *Multiscale connectivity based on a discrete-scale granulometry. The original microscopic image of an electronic circuit is thresholded and the resulting binary tracks are considered separately, leading to the binary images A_1, A_2, and A_3, corresponding to tracks 2, 6, and 8, respectively. Tracks 2 and 6 have nonzero degree of connectivity, whereas the degree of connectivity of track 8 is zero, since this track is fully disconnected. Track 6 has a higher degree of connectivity than track 2; i.e., track 6 is "more connected" than track 2.*

5 Second-Generation Multiscale Connectivities

In this section, we show that, starting from a given multiscale connectivity, we can construct a new multiscale connectivity by means of a clustering or an opening. Following the terminology used by J. Serra for the single-scale case [13], the resulting multiscale connectivity is referred to as a *second-generation multiscale connectivity.*

5.1 Second-Generation Multiscale Connectivities Based on Clusterings

We are interested here in operators that are clusterings at all levels of a connectivity pyramid. We refer to such operators as *full clusterings*. A full clustering is said to be strong if it is a strong clustering at each connectivity scale. It will soon become clear that full clusterings are *connectivity-increasing* operators. An operator ψ on $\mathcal{L}$ is connectivity-increasing if it always increases the degree of connectivity of an element; i.e., if $\varphi(\psi(A)) \geq \varphi(A)$, for all $A \in \mathcal{L}$. The following result shows that connectivity-increasing operators are the multiscale counterpart of connectivity-preserving operators.

5.1 Proposition. Let $\mathcal{L}$ be a lattice, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. An operator ψ on $\mathcal{L}$ is connectivity-preserving at each scale $\sigma \in \Sigma$ if and only if it is connectivity-increasing.

Proof. To show the direct implication, let $A \in \mathcal{L}$. If $\varphi(A) = O_\Sigma$, then $\varphi(\psi(A)) \geq \varphi(A)$ and we are done. Otherwise, we have that $A \in \mathbf{C}(\varphi(A)) \Rightarrow \psi(A) \in \mathbf{C}(\varphi(A)) \Rightarrow \varphi(\psi(A)) \geq \varphi(A)$. We now show the reverse implication. For a given $\sigma \in \Sigma_0$ and $C \in \mathbf{C}(\sigma)$, we have that $\varphi(\psi(C)) \geq \varphi(C) \geq \sigma \Rightarrow \psi(C) \in \mathbf{C}(\sigma)$, so that $\psi(\mathbf{C}(\sigma)) \subseteq \mathbf{C}(\sigma)$. Q.E.D.

Since full clusterings are connectivity-preserving at each connectivity scale, they are connectivity-increasing. Note that σ-connectivity openings are connectivity-increasing, since it can be easily verified that they are connectivity-preserving at all scales. Merging Props. 3.3 and 5.1, we conclude that a closing and an extensive dilation are full clusterings if and only if they are connectivity-increasing. In addition, the strong property holds for the case of dilations on infinite $\vee$-distributive lattices. Since for a dilation δ connectivity-preservation follows from the connectivity property of the sup-generators, it follows that δ is connectivity-increasing if and only $\delta(x)$ is fully connected, for each $x \in \mathcal{S}$. In fact, we have the following characterization of full clusterings.

5.2 Proposition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. An operator ψ on $\mathcal{L}$ is a full clustering if and only if:

(*i*) ψ is increasing and extensive.

(*ii*) ψ is connectivity-increasing.

(*iii*) For a family $\{A_\alpha\}$ in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$, we have $\varphi(\psi(\bigvee A_\alpha)) \geq \bigwedge \varphi(\psi(A_\alpha))$.

Proof. Properties (*i*) and (*ii*) above follow from conditions (*i*) and (*ii*) of Defn. 3.2 and Prop. 5.1. Let $\{A_\alpha\}$ be a family in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$. We now show that condition (*iii*) of Defn. 3.2 holds at each connectivity scale if and only if property (*iii*) above holds. To show the direct implication, let $\sigma_0 = \bigwedge \varphi(\psi(A_\alpha))$. If $\sigma_0 = O_\Sigma$, there is nothing to prove. Otherwise, we have that $\varphi(\psi(A_\alpha)) \geq \sigma_0 \Rightarrow \psi(A_\alpha) \in \mathbf{C}(\sigma_0),\ \forall\, \alpha \Rightarrow \psi(\bigvee A_\alpha) \in \mathbf{C}(\sigma_0) \Rightarrow \varphi(\psi(\bigvee A_\alpha)) \geq \sigma_0 = \bigwedge \varphi(\psi(A_\alpha))$, as required. To show the reverse implication, note that, given $\sigma \in \Sigma_0$, we have $\psi(A_\alpha) \in \mathbf{C}(\sigma),\ \forall \alpha \Rightarrow \varphi(\psi(A_\alpha)) \geq \sigma,\ \forall \alpha \Rightarrow \bigwedge \varphi(\psi(A_\alpha)) \geq \sigma$, so that $\varphi(\psi(\bigvee A_\alpha)) \geq \bigwedge \varphi(\psi(A_\alpha)) \geq \sigma \Rightarrow \psi(\bigvee A_\alpha) \in \mathbf{C}(\sigma)$, as required. Q.E.D.

The following proposition shows that, starting from a given multiscale connectivity system, a multiscale clustering generates a new multiscale connectivity system that expands the original one.

5.3 Proposition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$, and let ψ be a full clustering on $\mathcal{L}$. We have that:

(a) $\varphi^\psi\colon \mathcal{L} \to \Sigma$, given by $\varphi^\psi = \varphi(\psi(\cdot))$, is a connectivity measure on $\mathcal{L}$, such that $\varphi \leq \varphi^\psi$.

(b) $\mathbf{C}^\psi\colon \Sigma_0 \to \mathcal{P}(\mathcal{L})$, given by $\mathbf{C}^\psi(\sigma) = \psi^{-1}(\mathbf{C}(\sigma)) = \{A \in \mathcal{L} \mid \psi(A) \in \mathbf{C}(\sigma)\}$, for $\sigma \in \Sigma_0$, is a connectivity pyramid on $\mathcal{L}$, such that $\mathbf{C} \leq \mathbf{C}^\psi$.

(c) $(\varphi^\psi, \mathbf{C}^\psi)$ constitute a multiscale connectivity system on $\mathcal{L}$.

PROOF. (a) Note that $\varphi^\psi(O) = \varphi(\psi(O)) \geq \varphi(O) = I_\Sigma \Rightarrow \varphi^\psi(O) = I_\Sigma$, since ψ is connectivity-increasing. By using the same argument, we can show that $\varphi^\psi(x) = I_\Sigma$, for $x \in \mathcal{S}$. This shows property (*i*) of a connectivity measure. Property (*ii*) of a connectivity measure follows directly from item (*iii*) of Proposition 5.2. The inequality $\varphi \leq \varphi^\psi$ is a direct consequence of the fact that ψ is connectivity-increasing, since $\varphi^\psi(A) = \varphi(\psi(A)) \geq \varphi(A)$, for all $A \in \mathcal{L}$.

(b) Let Γ be the mapping from $\mathcal{M}(\mathcal{L}, \Sigma)$ into $\mathcal{Y}(\mathcal{L}, \Sigma)$, given by (1). From part (a), we have that $\varphi^\psi \in \mathcal{M}(\mathcal{L}, \Sigma)$. Now, $\Gamma(\varphi^\psi)(\sigma) = \{A \in \mathcal{L} \mid \varphi^\psi(A) \geq \sigma\} = \{A \in \mathcal{L} \mid \varphi(\psi(A)) \geq \sigma\} = \{A \in \mathcal{L} \mid \psi(A) \in \mathbf{C}(\sigma)\} = \mathbf{C}^\psi(\sigma)$, for $\sigma \in \Sigma_0$. Therefore, $\mathbf{C}^\psi \in \mathcal{Y}(\mathcal{L}, \Sigma)$; i.e., $\mathbf{C}^\psi$ is a connectivity pyramid on $\mathcal{L}$. The inequality $\mathbf{C} \leq \mathbf{C}^\psi$ follows from $\varphi \leq \varphi^\psi$ and the fact that Γ is order-preserving.

(c) This is a direct consequence of the argument used in the proof of part (b). Q.E.D.

As a consequence of the fact that $\mathbf{C} \leq \mathbf{C}^\psi$, it follows that, for $A \in \mathcal{L}$, the HPCC $\mathbf{c}_A^\psi$ of A according to $(\varphi^\psi, \mathbf{C}^\psi)$ is *coarser* than the HPCC $\mathbf{c}_A$ of A according to $(\varphi, \mathbf{C})$; i.e.,

$$\mathbf{c}_A(\sigma, x) = \gamma_{\sigma,x}(A) \leq \gamma_{\sigma,x}^\psi(A) = \mathbf{c}_A^\psi(\sigma, x), \quad \sigma \in \Sigma_0,\ x \in \mathcal{S}(A),$$

where $\{\gamma_{\sigma,x} \mid \sigma \in \Sigma_0, x \in \mathcal{S}\}$ and $\{\gamma_{\sigma,x}^\psi \mid \sigma \in \Sigma_0, x \in \mathcal{S}\}$ are the σ-connectivity openings associated with $(\varphi, \mathbf{C})$ and $(\varphi^\psi, \mathbf{C}^\psi)$, respectively.

The multiscale connectivity system $(\varphi^\psi, \mathbf{C}^\psi)$ of Proposition 5.3 is said to produce a *second-generation multiscale connectivity based on clustering.* This multiscale connectivity is “larger” than the original one, in the sense that every element has a higher degree of connectivity, there are more σ-connected elements at each scale σ, and the HPCC is coarser.

If the full clustering is strong, it follows from the corresponding single-scale result [2, Proporsition 6.6] that the σ-connectivity openings $\{\gamma^{\psi}_{\sigma,x} \mid \sigma \in \Sigma_0,\ x \in \mathcal{S}\}$, associated with $(\varphi^{\psi}, \mathbf{C}^{\psi})$, are given by

$$\gamma^{\psi}_{\sigma,x}(A) = \begin{cases} A \wedge \gamma_{\sigma,x}\psi(A), & \text{if } \ x \leq A \\ O, & \text{if } \ x \not\leq A \end{cases}, \quad \sigma \in \Sigma_0,\ A \in \mathcal{L},\ x \in \mathcal{S}\,,$$

where $\{\gamma_{\sigma,x} \mid \sigma \in \Sigma_0,\ x \in \mathcal{S}\}$ are the σ-connectivity openings associated with the original multiscale connectivity system $(\varphi, \mathbf{C})$. If in addition $\mathcal{L}$ is infinite $\vee$-distributive, the σ-reconstruction operators $\{\rho^{\psi}_{\sigma} \mid \sigma \in \Sigma_0\}$ associated with $(\varphi^{\psi}, \mathbf{C}^{\psi})$ are given by

$$\rho^{\psi}_{\sigma}(A \mid M) = A \ \wedge\ \rho_{\sigma}(\psi(A) \mid A \wedge M), \quad \sigma \in \Sigma_0,\ A, M \in \mathcal{L}\,,$$

where $\{\rho_{\sigma} \mid \sigma \in \Sigma_0\}$ is the σ-reconstruction operator associated with $(\varphi, \mathbf{C})$.

5.2 Second-Generation Multiscale Connectivities Based on Openings

Another way to construct a multiscale connectivity from an existing one is by means of an opening. The following results hold if contractions (see Section 4.1) are used instead of openings [5].

5.4 Proposition. Consider an atomic lattice $\mathcal{L}$ with sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. Let θ be an opening on $\mathcal{L}$. The mapping $\mathbf{C}^{\theta}\colon\ \Sigma_0 \to \mathcal{P}(\mathcal{L})$, given by

$$\mathbf{C}^{\theta}(\sigma) = \{O\} \cup \mathcal{S} \cup \{A \in \mathbf{C}(\sigma) \mid \theta(A) = A\}, \quad \sigma \in \Sigma_0, \tag{21}$$

is a connectivity pyramid on $\mathcal{L}$, such that $\mathbf{C}^{\theta} \leq \mathbf{C}$, with associated connectivity measure $\varphi^{\theta}\colon\ \mathcal{L} \to \Sigma$, given by

$$\varphi^{\theta}(A) = \begin{cases} I_{\Sigma}, & \text{if } A = O \text{ or } A \in \mathcal{S} \\ \varphi(A), & \text{if } \theta(A) = A \\ O_{\Sigma}, & \text{otherwise} \end{cases}, \quad A \in \mathcal{L}, \tag{22}$$

such that $\varphi^{\theta} \leq \varphi$.

Proof. From the single-scale result [2, Proposition 7.1], we have that $\mathbf{C}^{\theta}(\sigma)$ is a connectivity class in $\mathcal{L}$, for each $\sigma \in \Sigma_0$, which shows property *(i)* of a connectivity pyramid. Clearly, $\mathbf{C}(\sigma) \subseteq \mathbf{C}(\tau)$ implies that $\mathbf{C}^{\theta}(\sigma) \subseteq \mathbf{C}^{\theta}(\tau)$, for $\sigma \geq \tau$, which shows property *(ii)* of a connectivity pyramid. We now show property *(iii)* of a connectivity pyramid. Given a nonempty set of scales $\{\sigma_{\alpha}\} \subseteq \Sigma_0$, we have that $\mathbf{C}^\theta(\bigvee \sigma_\alpha) = \{O\} \cup \mathcal{S} \cup [\mathbf{C}(\bigvee \sigma_\alpha) \cap \mathrm{Inv}(\theta)] = \{O\} \cup \mathcal{S} \cup [(\bigcap \mathbf{C}(\sigma_\alpha)) \cap \mathrm{Inv}(\theta)] = \{O\} \cup \mathcal{S} \cup \bigcap[\mathbf{C}(\sigma_\alpha) \cap \mathrm{Inv}(\theta)] = \bigcap(\{O\} \cup \mathcal{S} \cup [\mathbf{C}(\sigma_\alpha) \cap \mathrm{Inv}(\theta)]) = \bigcap \mathbf{C}^\theta(\sigma_\alpha)$, since $\mathcal{P}(\mathcal{L})$ is infinite $\wedge$-distributive. Moreover, it is obvious that $\mathbf{C}^\theta(\sigma) \subseteq \mathbf{C}(\sigma)$, for each $\sigma \in \Sigma_0$; i.e., $\mathbf{C}^\theta \leq \mathbf{C}$. Finally, (22) follows easily from (2) and (21), whereas the inequality $\varphi^\theta \leq \varphi$ is a direct consequence of (22), or of the fact that $\mathbf{C}^\theta \leq \mathbf{C}$. Q.E.D.

The multiscale connectivity system $(\varphi^\theta, \mathbf{C}^\theta)$ of Proposition 5.4 is said to produce a *second-generation multiscale connectivity based on opening.* Note that this second generation connectivity implies that all elements of $\mathcal{L}$ that are not invariant to θ become fully disconnected (this affords robustness against spurious random grains; however, spurious random pores can disconnect a big grain under this connectivity). Therefore, the new multiscale connectivity is "stricter" than the original one; i.e., every element has a lower degree of connectivity and there are fewer σ-connected elements at each scale σ.

An opening θ is said to be *fully locally invariant* if θ is locally invariant at each level of a connectivity pyramid. For example, if $(\varphi, \mathbf{C})$ is a translation-invariant multiscale connectivity system on $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$, and B is a fully connected structuring element, then it is easy to check that the structural opening $\theta_B(A) = A \circ B$ is fully locally invariant. In the case of fully locally invariant opening θ, it follows from the corresponding single-scale result [2, Proporsition 7.5] that the σ-connectivity openings $\{\gamma^\theta_{\sigma,x} \mid \sigma \in \Sigma_0,\ x \in \mathcal{S}\}$, associated with $(\varphi^\theta, \mathbf{C}^\theta)$, are given by

$$\gamma^\theta_{\sigma,x}(A) = \begin{cases} \gamma_{\sigma,x}\theta(A), & \text{if } \ x \leq \theta(A) \\ x, & \text{if } \ \theta(A) \not\geq x \leq A \\ O, & \text{if } \ x \not\leq A \end{cases}, \quad \sigma \in \Sigma_0,\ A \in \mathcal{L},\ x \in \mathcal{S}.$$

where $\{\gamma_{\sigma,x} \mid \sigma \in \Sigma_0,\ x \in \mathcal{S}\}$ are the σ-connectivity openings associated with the original multiscale connectivity system $(\varphi, \mathbf{C})$. If in addition $\mathcal{L}$ is infinite $\vee$-distributive, the σ-reconstruction operators $\{\rho^\theta_\sigma \mid \sigma \in \Sigma_0\}$ associated with $(\varphi^\theta, \mathbf{C}^\theta)$ are given by

$$\rho^\theta_\sigma(A \mid M) = (A \wedge M) \vee \rho_\sigma(\theta(A) \mid M), \quad \sigma \in \Sigma_0,\ A, M \in \mathcal{L}.$$

where $\{\rho_\sigma \mid \sigma \in \Sigma_0\}$ is the σ-reconstruction operator associated with $(\varphi, \mathbf{C})$.

It should be noted that new examples of multiscale connectivity can be obtained with the techniques presented in this section. For example, a clustering may be applied to a multiscale connectivity based on openings, discussed in Section 4, to generate a new example of second-generation multiscale connectivity. By the same token, an opening may be applied to a multiscale connectivity based on pyramids of clusterings, discussed in Section 3. The generality of the approach resides in the fact that *any* given multiscale connectivity may be used as the basis for constructing a new second-generation example, by using either clusterings or openings. Construction of second generation connectivities that may be useful in image analysis applications is left as a future research endeavor.

6 Conclusion

In this paper, we have suggested three general techniques for constructing multiscale connectivities. The first two techniques start from a given connectivity class and produce multiscale connectivities by means of pyramids of clusterings or granulometries. The third technique produces second-generation multiscale connectivities, starting from a given base multiscale connectivity. We have presented several examples of multiscale connectivities based on dilation, closing, and opening pyramids, including an example based on the theory of grayscale level connectivity and another example based on the theory of morphological sampling. It is expected that multiscale connectivities constructed with the techniques presented in this paper will be useful in several image processing and analysis applications.

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