

Multiscale Connected Operators

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Abstract: Among the major developments in Mathematical Morphology in the last two decades are the interrelated subjects of connectivity classes and connected operators. Braga-Neto and Goutsias have proposed an extension of the theory of connectivity classes to a multiscale setting, whereby one can assign connectivity to an object observed at different scales. In this paper, we study connected operators in the context of multiscale connectivity. We propose the notion of a σ -connected operator, that is, an operator connected at scale σ . We devote some attention to the study of binary σ -grain operators. In particular, we show that families of σ -grain openings and σ -grain closings, indexed by the connectivity scale parameter, are granulometries and anti-granulometries, respectively. We demonstrate the use of multiscale connected operators with image analysis applications. The first is the scale-space representation of grayscale images using multiscale levelings, where the role of scale is played by the connectivity scale. Then we discuss the application of multiscale connected openings in granulometric analysis, where both size and connectivity information are summarized. Finally, we describe an application of multiscale connected operators to an automatic target recognition problem.

Keywords: Connectivity, Connectivity Classes, Connected Operators, Multiscale Connectivity, Grain Operators, Flattenings, Levelings, Image Analysis, Scale-Space, Granulometry, Pattern Spectrum, Granulometric Analysis, Automatic Target Recognition.

1 Introduction

Among the major developments in Mathematical Morphology in the last two decades are the interrelated subjects of *connectivity classes*, first proposed for Boolean lattices by J. Serra and G. Matheron in the late eighties [1] (and later extended to general complete lattices by J. Serra [2, 3]), and *connected operators*, which have been studied by J. Serra, P. Salembier, J. Crespo, H. Heijmans and F. Meyer [4–10], among others. This has represented an important advance in the field of image analysis, since connectivity, as well as *object-based methods* implemented by connected operators, play a prominent role in problems related to image filtering and segmentation, image compression and coding, motion analysis, pattern recognition, and other applications.

Connectivity classes, in their general lattice-theoretical formulation, not only unify in a single axiomatic framework most useful notions of connectivity, such as topological connectivity, graph-theoretic connectivity, fuzzy connectivity, etc., but also include new interesting definitions of connectivity [2, 3, 11–13]. *Multiscale connectivity* was introduced in [14, 15], as a general multiscale theory of connectivity on complete lattices, which extended the theory of connectivity classes. The idea of multiscale connectivity arises naturally from the observation that the connectivity of an object may depend on the particular scale at which it is observed. Dependence of connectivity on scale can be equivalently characterized by either a *connectivity measure*, which quantifies the degree of connectivity of an object, or by a *connectivity pyramid*, which is a nested sequence of connectivity classes that depend on scale.

Connected operators, on the other hand, have become very popular in recent years for image analysis applications [16–19]. This is mainly due to the fact that these operators do not work at the pixel level, but rather at the level of the flat zones of an image, which are defined using connectivity criteria. A connected operator can remove boundaries, but cannot shift boundaries or introduce new ones. It therefore preserves contour/shape information, known to carry most of image content perceived by human observers. Connected operators are well-suited to development of *object-based* image analysis algorithms [20, 21].

In this paper, we propose to study connected operators in the context of multiscale connectivity. After a review of aspects of connectivity theory needed for this study, we define the notion of a σ -*connected operator*, that is, an operator connected at scale σ . We devote some attention to the study of binary σ -*grain operators*, which extend the special class of connected operators known as *grain operators* [7, 8], also known as set *flattenings* and *levelings* [3]. We show that families of σ -grain openings and σ -grain closings, indexed by the connectivity scale parameter, are granulometries and anti-granulometries,

respectively. Note that the notion of scale associated with usual granulometries of connected operators are typically related to *size* [22], whereas here the granulometries are associated with the connectivity scale. The two kinds of information may be combined in multiscale representation of images.

We demonstrate the use of multiscale connected operators with image analysis applications. The first is the scale-space representation of grayscale images using multiscale levelings, where the role of scale is played by the connectivity scale. Then we discuss the application of multiscale connected openings in granulometric analysis, where both size and connectivity information are summarized. Finally, we describe an application of multiscale connected operators to an automatic target recognition problem.

2 Connectivity Theory

In this section, we review aspects of connectivity theory, such as connectivity classes, connected operators and multiscale connectivity. Our intention is to make the presentation self-contained, by summarizing the main concepts and results that are relevant to our purposes. For detailed treatment, the reader is referred to [1, 2, 13] (connectivity classes), [5, 8] (connected operators), and [15] (multiscale connectivity). We consider in this paper connectivity on Boolean lattices, which is sufficient for the development of both binary and multivalued (e.g., grayscale) connected operators. Please see the aforementioned references for the general lattice-theoretical formulation of connectivity classes and multiscale connectivity.

2.1 Connectivity Classes

The major component of connectivity theory is the notion of connectivity classes, introduced for Boolean lattices by G. Matheron and J. Serra in the late eighties [1].

2.1 Definition. A family $\mathcal{C} \subseteq \mathcal{P}(E)$ is called a *connectivity class* if the following conditions are satisfied:

- (i) $\emptyset \in \mathcal{C}$,
- (ii) $\{x\} \in \mathcal{C}$, for all $x \in E$,
- (iii) for a family $\{C_\alpha\}$ in \mathcal{C} such that $\bigcap C_\alpha \neq \emptyset$, we have that $\bigcup C_\alpha \in \mathcal{C}$.

The elements of \mathcal{C} are said to be *connected*.

These conditions require that the empty set and the points be connected, and that the union of overlapping connected sets be connected. These are considered to be a minimal set of requirements

that must be satisfied by connectivity; in fact, they are the minimum set of properties that lead to partitioning of a set into disjoint connected components [2, 13].

We say that C is a *connected component* of $A \in \mathcal{P}(E)$, which we denote by $C \triangleleft A$, if $C \in \mathcal{C}$, $C \subseteq A$ and there is no $C' \in \mathcal{C}$ different from C such that $C \subseteq C' \subseteq A$. In other words, a connected component of an object is a maximal connected part of the object. The set of connected components of A is denoted by $\mathcal{C}(A)$. The elements of $\mathcal{C}(A)$ and $\mathcal{C}(A^c)$ are called the *grains* and *pores* of A , respectively.

One can define operators that extract connected components, known as *connectivity openings*:

$$\gamma_x(A) = \bigcup \{C \in \mathcal{C} \mid x \in C \subseteq A\}, \quad x \in E, A \in \mathcal{P}(E). \quad (1)$$

Clearly, these operators are *openings* (i.e., increasing, anti-extensive, and idempotent operators [23]). It is also easy to see that $\gamma_x(A)$ is the connected component C of A that is *marked* by x (i.e., such that $x \in C$). Moreover, any connected component of A can be extracted by a connectivity opening γ_x , for some $x \in E$. Hence, the connectivity openings completely characterize the connected components of A . Moreover, a fundamental result of connectivity theory is that the family of connectivity openings $\{\gamma_x \mid x \in E\}$ uniquely characterizes the connectivity class \mathcal{C} with which it is associated [1, 11]. The dual operator to the connectivity opening is the *connectivity closing*, given by

$$\varphi_x(A) = \gamma_x^*(A) = (\gamma_x(A^c))^c = A \cup \bigcup \{C \in \mathcal{C} \mid x \notin C \subseteq A^c\}, \quad x \in E, A \in \mathcal{P}(E), \quad (2)$$

which fills up all pores of A except one marked by x .

For a marker $M \in \mathcal{P}(E)$, the *reconstruction* $\rho(A \mid M)$ of a given $A \in \mathcal{P}(E)$ from M is defined by

$$\rho(A \mid M) = \bigcup_{x \in M} \gamma_x(A) = \bigcup \{C \in \mathcal{C}(A) \mid C \cap M \neq \emptyset\}. \quad (3)$$

Hence, the reconstruction operator $\rho(A \mid M)$ extracts the grains of A that intersect marker M . It follows from the fact that the supremum of openings is an opening [23] that the operator $\rho(\cdot \mid M)$ is an opening on $\mathcal{P}(E)$. Like connectivity openings, the reconstruction operator ρ characterizes, in a unique fashion, the connectivity class with which it is associated. This fact was established for the lattice $\mathcal{P}(E)$ in [8] (it has been extended to connectivity classes in infinite \vee -distributive lattices in [12], and independently in [24]). The *dual reconstruction* $\xi(A \mid M)$ of a given $A \in \mathcal{P}(E)$ from a marker $M \in \mathcal{P}(E)$ is defined by:

$$\begin{aligned} \xi(A \mid M) &= \rho^*(A \mid M) = (\rho(A^c \mid M^c))^c = \bigcap_{x \in M^c} \varphi_x(A) \\ &= A \cup \bigcup \{C \in \mathcal{C}(A^c) \mid C \cap M^c = \emptyset\} \\ &= A \cup \bigcup \{C \in \mathcal{C}(A^c) \mid C \subseteq M\}. \end{aligned} \quad (4)$$

Hence, the dual reconstruction fills up the pores of A that are covered by M . The operator $\xi(\cdot \mid M)$ is a closing on $\mathcal{P}(E)$.

2.2 Connected Operators

A partition π of a set A is defined in the usual fashion: the elements of π are disjoint subsets of A , whose union is A . A partition π of E is said to be *coarser* than another partition π' of E if every set in π' is contained in some set in π . The set $\mathcal{C}(A)$ is clearly a partition of A , called the *partition of connected components* (PCC) of A . The set $\mathcal{C}(A) \cup \mathcal{C}(A^c)$ of grains and pores of A is a partition of E , called the *partition of flat zones* (PFZ) of A .

More generally, let T^E be an image space consisting of functions from E into a value set T (set T is arbitrary, except where noted). The partition of flat zones of an image $f \in T^E$ is defined as the set of maximal connected subsets of E over which the image is constant, according to a connectivity class in $\mathcal{P}(E)$. If the value set is binary, then the two previous definitions of PFZ coincide, provided we identify a binary image with its support. Hence, we can treat both cases uniformly.

2.2 Definition. An operator ψ on T^E is said to be connected if, given any image $f \in T^E$, the PFZ of $\psi(f)$ is coarser than the PFZ of f , according to a given connectivity class in $\mathcal{P}(E)$.

Hence, an operator ψ is connected if, and only if, for any image f , the output $\psi(f)$ is constant over any flat zone of f . This means that a connected operator can only remove boundaries; it cannot shift, break, or introduce new boundaries (here, “boundary” is understood as the interface between flat zones). In the binary case, a connected operator can only remove entire grains, or fill entire pores. This is illustrated in Fig. 1.

The definition of a connected operator depends on the assumed connectivity class \mathcal{C} , which determines the partition of flat zones. Hence, ψ may be connected according to a given connectivity class, but not according to another. However, we have the following proposition (a specialization of this result to the binary case appears in [8]).

2.3 Proposition. Let \mathcal{C} and \mathcal{C}' be two connectivity classes in $\mathcal{P}(E)$, such that $\mathcal{C} \subseteq \mathcal{C}'$. Every connected operator on T^E according to \mathcal{C}' is also connected according to \mathcal{C} .

PROOF. Let Z be a flat zone of an image $f \in T^E$, according to \mathcal{C} . Since $\mathcal{C} \subseteq \mathcal{C}'$, the flat zone Z must be contained in some flat zone Z' according to \mathcal{C}' . If ψ is a connected operator according to \mathcal{C}' , then

$\psi(f)$ is constant over Z' , and thus it is constant over Z , as well. Hence, ψ is connected according to \mathcal{C} . Q.E.D.

The following result lists ways for creating new connected operators from existing ones (similar results for the binary case appear in [8]).

2.4 Proposition. Let ψ , ϕ , and $\{\psi_i \mid i \in I\}$ be connected operators on T^E .

(a) The composition $\psi\phi$ is connected.

(b) The supremum $\bigvee \psi_i$ and infimum $\bigwedge \psi_i$ are connected. □

PROOF. (a): The relation “coarser than” for partitions is clearly transitive. Since, for any image $f \in T^E$, the PFZ of $\psi\phi(f) = \psi[\phi(f)]$ is coarser than the PFZ of $\phi(f)$, which is coarser than the PFZ of f , it follows that the PFZ of $\psi\phi(f)$ is coarser than the PFZ of f .

(b): For any image $f \in T^E$, $\psi_i(f)$ is constant over any given flat zone Z of f , for all $i \in I$. Hence, $(\bigvee \psi_i)(f) = \bigvee \psi_i(f)$ also assumes a constant value over Z . An analogous argument holds for $\bigwedge \psi_i$. Q.E.D.

Connectivity openings, connectivity closings, reconstruction, and its dual, discussed in Section 2.1, are clearly connected operators on $\mathcal{P}(E)$. Other useful examples of connected operators in image analysis include the area opening and the opening by reconstruction, and their grayscale counterparts [23, 25, 26].

Our discussion of grain operators is similar to that in [8]. We define *foreground criteria* and *background criteria* as mappings $u, v : \mathcal{P}(E) \rightarrow \{0, 1\}$, respectively. If $u(A) = 1$ (resp. $v(A^c) = 1$), then we say that A satisfies the foreground (resp. background) criterion imposed by u (resp. v). Using this concept, we define the *foreground trivial operator* and the *background trivial operator* on $\mathcal{P}(E)$ by

$$\iota_u(A) = \begin{cases} A, & \text{if } u(A) = 1 \\ \emptyset, & \text{otherwise} \end{cases}, \quad (5)$$

$$\kappa_v(A) = \begin{cases} A, & \text{if } v(A^c) = 1 \\ E, & \text{otherwise} \end{cases}, \quad (6)$$

for $A \in \mathcal{P}(E)$. Note that these operators are dual to each other. Note also that ι_u and κ_v are connected operators, regardless of the assumed connectivity class.

Now, let \mathcal{C} be a connectivity class in $\mathcal{P}(E)$. The trivial operators allow us to define the *foreground grain operator* and the *background grain operator* by

$$\psi_u = \bigvee_{x \in E} \iota_u \gamma_x \quad (7)$$

$$\phi_v = \bigwedge_{x \in E} \kappa_v \varphi_x, \quad (8)$$

where $\{\gamma_x \mid x \in E\}$ and $\{\varphi_x \mid x \in E\}$ are the connectivity openings and connectivity closings associated with \mathcal{C} . Note that $\psi_u^* = (\bigvee_{x \in E} \iota_u \gamma_x)^* = \bigwedge_{x \in E} (\iota_u \gamma_x)^* = \bigwedge_{x \in E} \iota_u^* \gamma_x^* = \bigwedge_{x \in E} \kappa_u \varphi_x = \phi_u$; i.e., the foreground grain operator and the background grain operator are dual to each other. From the previous discussion, it becomes clear that, for $A \in \mathcal{P}(E)$,

$$\psi_u(A) = \bigcup \{C \leq A \mid u(C) = 1\} \quad (9)$$

$$\phi_v(A) = A \cup \bigcup \{C \leq A^c \mid v(C) = 0\}. \quad (10)$$

In other words, ψ_u keeps the grains that satisfy criterion u , whereas ϕ_v fills up the pores that do not satisfy criterion v .

We can see from (9) and (10) that ψ_u and ϕ_v are anti-extensive and extensive connected operators on $\mathcal{P}(E)$, respectively. Such operators can be characterized by means of a simple commutativity criterion (the following result was shown independently in [7]).

2.5 Proposition.

(a) An anti-extensive connected operator θ on $\mathcal{P}(E)$ is a foreground grain operator if and only if

$$\gamma_x \theta = \theta \gamma_x, \quad x \in E. \quad (11)$$

(b) An extensive connected operator θ on $\mathcal{P}(E)$ is a background grain operator if and only if

$$\varphi_x \theta = \theta \varphi_x, \quad x \in E. \quad (12)$$

PROOF. We show only part (a); part (b) follows by duality. The direct implication (only if) follows from the fact that $\gamma_x \psi_u(A) = \psi_u \gamma_x(A) = \iota_u \gamma_x(A)$, for $A \in \mathcal{P}(E)$. To show the reverse implication (if), assume that θ is an anti-extensive connected operator such that (11) holds. Note that (11) implies that $\theta(C) = C$ or \emptyset , for $C \in \mathcal{C}$. Define a foreground criterion u for $C \in \mathcal{C}$ as follows (the value of u outside \mathcal{C} is not relevant here):

$$u(C) = \begin{cases} 1, & \text{if } \theta(C) = C \\ 0, & \text{if } \theta(C) = \emptyset \end{cases}. \quad (13)$$

We show that $\theta = \psi_u$. First, note that $\theta \gamma_x = \psi_u \gamma_x$, since $\theta \gamma_x(A) = \gamma_x(A) = \psi_u \gamma_x(A)$, if $u(\gamma_x(A)) = 1$, and $\theta \gamma_x(A) = \emptyset = \psi_u \gamma_x(A)$, otherwise. From (11), it follows that $\theta = \bigvee_{x \in E} \gamma_x \theta = \bigvee_{x \in E} \theta \gamma_x = \bigvee_{x \in E} \psi_u \gamma_x = \psi_u$. Q.E.D.

As a straightforward corollary of Proposition 2.5, the output of foreground and background grain operators can be computed grain by grain and pore by pore, respectively.

2.6 Corollary. For every $A \in \mathcal{P}(E)$,

$$\psi_u(A) = \bigcup_{x \in E} \psi_u \gamma_x(A) \quad (14)$$

$$\phi_v(A) = \bigcap_{x \in E} \phi_v \varphi_x(A). \quad (15)$$

PROOF. From (11), we have that $\psi_u(A) = \bigcup_{x \in E} \gamma_x \psi_u(A) = \bigcup_{x \in E} \psi_u \gamma_x(A)$, which shows (14). Equation (15) follows by duality. Q.E.D.

Clearly, the trivial operators and the foreground and background grain operators are idempotent. Moreover, it is clear from (5) and (6) that if the criteria u and v are increasing, then ι_u and κ_v are increasing (these become respectively the *trivial opening* and the *trivial closing* defined in [1]). It follows from (7) and (8) that ψ_u and ϕ_v are also increasing operators, in which case, ψ_u is a *grain opening*, while ϕ_v is a *grain closing*. The connectivity opening γ_x and the reconstruction operator $\rho(\cdot|M)$ are grain openings, associated respectively with the increasing criteria $u(C) = 1$ if $x \in C$ and $u'(C) = 1$ if $C \cap M \neq \emptyset$. Similarly, the connectivity closing φ_x and the dual reconstruction $\xi(\cdot|M)$ are grain closings, associated respectively with the increasing criteria $v(C) = 0$ if $x \notin C$ and $v'(C) = 0$ if $C \subseteq M$. Foreground and background grain operators can be designed with non-increasing criteria (e.g., criteria based on the perimeter), in which case the operators are still (anti-)extensive and idempotent, in addition to being connected.

Now, consider the following connected operator, which simultaneously acts on the foreground and the background:

$$\zeta_{u,v}(A) = \bigcup \{C \mid C \triangleleft A \text{ and } u(C) = 1 \text{ or } C \triangleleft A^c \text{ and } v(C) = 0\}, \quad A \in \mathcal{P}(E), \quad (16)$$

where, as before, u and v are foreground and background criteria. This is called the *grain operator* in [8]. It is called a *set flattening* in [3]. Grain operators clearly include as special cases the foreground and background grain operators considered previously, since $\psi_u = \zeta_{u,1}$ and $\phi_v = \zeta_{1,v}$.

Two connected components C_1 and C_2 are said to be *adjacent* if $C_1 \cup C_2 \in \mathcal{C}$. A connected operator is said to be *stable* if it does not swap the values of any grain and an adjacent pore [7, 8], so it preserves the orientation of edges. Clearly, a grain operator $\zeta_{u,v}$ is stable if and only if $u(C_1) \vee v(C_2) = 1$ for any grain C_1 adjacent to a pore C_2 in any $A \in \mathcal{P}(E)$. A stable grain operator corresponds to a *set leveling*, in the terminology of [3]. Clearly, foreground and background operators and, in particular, grain openings and grain closings, are always set levelings (since one of the criteria is identically one).

It is not true in general that $\zeta_{u,v} = \psi_u \phi_v$. Rather, by using equations (9) and (10), it is easy to see

that

$$\zeta_{u,v}(A) = \psi_u(A) \cup [A^c \cap \phi_v(A)] = [\psi_u(A) \cup A^c] \cap \phi_v(A), \quad A \in \mathcal{P}(E). \quad (17)$$

However, if u and v are increasing and $\zeta_{u,v}$ is increasing, it is shown in [8] that $\zeta_{u,v} = \psi_u \phi_v = \phi_v \psi_u$ (so that $\zeta_{u,v}$ is a strong filter, see [1]) if and only if $\zeta_{u,v}$ is a stable grain operator, i.e., a set leveling.

Grain operators of the form $\zeta_{u,v}$ are the only class of connected operators that act independently on each connected component of the foreground and the background, so that the output can be computed grain by grain and pore by pore (this is not true for all binary connected operators). The following result, which can be considered to be the analog of Proposition 2.5, makes this precise (this result appears in a slightly different form in [8]). To make the proof compact, we write $A(Z) = t$ to indicate that a set A is constant over a set Z , where $t = 1$ if $Z \subseteq A$, while $t = 0$ if $Z \subseteq A^c$.

2.7 Proposition. A connected operator ψ is a grain operator if and only if, for any $A_1, A_2 \in \mathcal{P}(E)$ such that C is a flat zone of both A_1 and A_2 , then $\psi(A_1)(C) = \psi(A_2)(C)$.

PROOF. The direct implication (only if) is straightforward. To show the reverse implication (if), let $u(C) = \psi(C)(C)$ and $v(C) = 1 - \psi(C^c)(C)$, for $C \in \mathcal{C}$ (the value of u and v outside \mathcal{C} does not matter here). It suffices to show that $\psi(A)(C) = \zeta_{u,v}(A)(C)$, for all $A \in \mathcal{P}(E)$, where C is any flat zone (grain or pore) of A . To fix ideas, assume that C is a grain of A . Then, $\zeta_{u,v}(A) = u(C) = \psi(C)(C)$. But C is a grain of both A and C . Thus, by hypothesis (with $A_1 = A$ and $A_2 = C$), we have that $\psi(A)(C) = \psi(C)(C)$. Therefore, $\psi(A)(C) = \zeta_{u,v}(A)(C)$, as required. The argument for the case when C is a pore of A is completely analogous. Q.E.D.

In other words, a connected operator ψ is a grain operator if and only if, for each flat zone C of A , we have that $\psi(A)(C)$ depends only on C (the criterion in Proposition 2.7 was actually adopted as the definition of a grain operator in [7]).

2.3 Multiscale Connectivity

In this section, we provide an outline of the theory of multiscale connectivity [15]. Here, we specialize it to the Boolean-lattice case. The set of scales for multiscale analysis is taken to be a general lattice Σ . The greatest and least elements of Σ are denoted by I_Σ and O_Σ , respectively (in this paper, “lattice” means “complete lattice”, and we refer to “lattice \mathcal{L} ,” without explicit reference to the underlying partial order; see [23, 27] for a detailed treatment of lattice theory).

The following definition introduces the concept of a connectivity measure on $\mathcal{P}(E)$.

2.8 Definition. A function $\varphi: \mathcal{P}(E) \rightarrow \Sigma$ is a *connectivity measure* if:

- (i) $\varphi(\emptyset) = \varphi(\{x\}) = I_\Sigma$, for $x \in E$,
- (ii) for a family $\{A_\alpha\}$ in $\mathcal{P}(E)$ such that $\bigcap A_\alpha \neq \emptyset$, we have that $\varphi(\bigcup A_\alpha) \geq \bigwedge \varphi(A_\alpha)$.

Given $A \in \mathcal{P}(E)$, the value $\varphi(A)$ indicates the degree of connectivity of A . Condition (i) of a connectivity measure requires the empty set and the points to be fully connected, whereas condition (ii) requires that the degree of connectivity of the union of overlapping sets must be greater than or equal to the “smallest” degree of connectivity of the individual sets.

Given a connectivity class \mathcal{C} , it is easy to verify that $\varphi: \mathcal{P}(E) \rightarrow \{0, 1\}$, given by $\varphi(A) = 1$, if $A \in \mathcal{C}$, and $\varphi(A) = 0$, if $A \notin \mathcal{C}$, is a connectivity measure. In other words, each $A \in \mathcal{L}$ is either fully connected, if $A \in \mathcal{C}$, or fully disconnected, if $A \notin \mathcal{C}$. Hence, connectivity classes correspond to *single-scale* connectivities, where the degree of connectivity is all-or-nothing; i.e., there is no intermediate connectivity.

A concept intimately related to connectivity measures is that of a connectivity pyramid. In what follows, $\Sigma_0 = \Sigma \setminus \{O_\Sigma\}$.

2.9 Definition. A mapping $\mathbf{C}: \Sigma_0 \rightarrow \mathcal{P}(\mathcal{P}(E))$ is a *connectivity pyramid* if:

- (i) $\mathbf{C}(\sigma)$ is a connectivity class, for each $\sigma \in \Sigma_0$,
- (ii) $\mathbf{C}(\sigma) \subseteq \mathbf{C}(\tau)$, for $\sigma \geq \tau$,
- (iii) $\mathbf{C}(\bigvee \sigma_\alpha) = \bigcap \mathbf{C}(\sigma_\alpha)$, for all nonempty set of scales $\{\sigma_\alpha\} \subseteq \Sigma_0$.

Therefore, a connectivity pyramid is a nested sequence of connectivity classes that satisfy an upper semi-continuity condition. The connectivity class $\mathbf{C}(\sigma)$ is said to be the σ -*level* or the σ -*connectivity class* of \mathbf{C} ; it may be thought of as the connectivity class assigned at scale σ . The nesting condition (ii) of a connectivity pyramid implies that fewer sets are connected as one moves upward in the pyramid (i.e., a connected set at a given level of the pyramid may not be connected at a higher level). In other words, more objects tend to be connected at small scales than at large scales. Note that condition (iii) actually implies condition (ii). However, both conditions are included in Definition 2.9 for clarity.

Given a connectivity class \mathcal{C} , we can define a simple connectivity pyramid \mathbf{C} on \mathcal{L} , by setting $\Sigma = \{0, 1\}$ and $\mathbf{C}(1) = \mathcal{C}$. Hence, as before, connectivity classes correspond to single-scale connectivities, where there is only one level of connectivity.

For $\sigma \in \Sigma_0$, C is σ -connected component or σ -grain of $A \in \mathcal{P}(E)$, which we denote by $C \triangleleft_\sigma A$, if it is a grain of A according to the connectivity class $\mathbf{C}(\sigma)$ at scale σ . Analogously, one can define a σ -pore of A .

In [15], it is shown that connectivity measures are equivalent to connectivity pyramids, in the sense that there is a one-to-one correspondence between them. The following is a specialization of Thm. 3.14 in [15].

2.10 Theorem. If φ is a connectivity measure, then

$$\mathbf{C}(\sigma) = \{A \in \mathcal{P}(E) \mid \varphi(A) \geq \sigma\}, \quad \sigma \in \Sigma_0, \quad (18)$$

is a connectivity pyramid. Conversely, if \mathbf{C} is a connectivity pyramid, then

$$\varphi(A) = \bigvee \{\sigma \in \Sigma_0 \mid A \in \mathbf{C}(\sigma)\}, \quad A \in \mathcal{P}(E). \quad (19)$$

is a connectivity measure. Moreover, the pair of equations (18) and (19) define a bijection between the set of all connectivity measures and the set of all connectivity pyramids.

Hence, to each connectivity measure φ there is a unique equivalent connectivity pyramid \mathbf{C} , and vice-versa. Therefore, we say that $\mathcal{P}(E)$ is furnished with a *multiscale connectivity system* (φ, \mathbf{C}) , such that φ and \mathbf{C} are equivalent under the bijection given in Theorem 2.10.

The family $\{\gamma_x^\sigma \mid \sigma \in \Sigma_0, x \in E\}$, where γ_x^σ is a connectivity opening associated with $\mathbf{C}(\sigma)$, are the σ -connectivity openings of a multiscale connectivity. The operation $\gamma_x^\sigma(A)$ extracts the σ -grain of A that is marked by x . As with the single-scale case, a family of σ -connectivity openings characterize uniquely the multiscale connectivity system with which it is associated (see [15] for the details). One can also define σ -connectivity closings $\varphi_x^\sigma(A) = (\gamma_x^\sigma)^*(A) = (\gamma_x^\sigma(A^c))^c$, which fills up all σ -pores of A except the one marked by x .

Similarly, the family $\{\rho^\sigma \mid \sigma \in \Sigma_0\}$, where ρ^σ is the reconstruction operator associated with $\mathbf{C}(\sigma)$, is the family of σ -reconstruction operators of a multiscale connectivity. This family characterizes uniquely the multiscale connectivity system with which it is associated, if the lattice is infinite \vee -distributive and Σ_0 is discrete [15]. The family of *dual σ -reconstruction operators* $\{\xi^\sigma \mid \sigma \in \Sigma_0\}$ is defined in the expected way.

An important practical fact about σ -reconstruction, which is a direct consequence of the nesting property of connectivity pyramids, is that, given a marker $M \in \mathcal{P}(E)$, the family $\{\rho^\sigma(\cdot | M) \mid \sigma \in \Sigma_0\}$ is a *granulometry*; i.e., $\rho^\sigma(A|M) \subseteq \rho^\tau(A|M)$, for $A \in \mathcal{P}(E)$ and $\sigma \geq \tau$. Similarly, the family of dual

σ -reconstruction operators $\{\xi^\sigma(\cdot|M) \mid \sigma \in \Sigma_0\}$ is an *anti-granulometry*; i.e., $\xi^\sigma(A|M) \supseteq \xi^\tau(A|M)$, for $A \in \mathcal{P}(E)$ and $\sigma \geq \tau$.

A simple example of multiscale connectivity is the following. Let $E = \mathbb{Z}^2$, with $\Sigma = \{0, 1, 2\}$. The mapping $\varphi : \mathcal{P}(\mathbb{Z}^2) \rightarrow \{0, 1, 2\}$, given by

$$\varphi(A) = \begin{cases} 2, & \text{if } A \text{ is 4-connected} \\ 1, & \text{if } A \text{ is 8- but not 4-connected,} \\ 0, & \text{otherwise} \end{cases}, \quad (20)$$

for $A \in \mathcal{P}(\mathbb{Z}^2)$, is a connectivity measure on $\mathcal{P}(\mathbb{Z}^2)$. The associated connectivity pyramid $\mathbf{C} : \{1, 2\} \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{Z}^2))$ is given by $\mathbf{C}(1) = \{8\text{-connected subsets of } \mathbb{Z}^2\}$ and $\mathbf{C}(2) = \{4\text{-connected subsets of } \mathbb{Z}^2\}$.

Next we discuss another example of multiscale connectivity, which will be used extensively in Section 4 to illustrate the use of multiscale connected operators in image analysis applications. It is based on extending a given connectivity by means of a family of dilations (interesting examples can be also defined using openings and closings, in addition to dilation. Please see [15, 28] for a general study of such multiscale connectivities).

Let $E = \mathbb{R}^2$ or $E = \mathbb{Z}^2$, with $\Sigma = \{-R, -(R-1), \dots, 0\}$, where R is some positive integer. Let \mathcal{C} be a *translation-invariant* connectivity class in $\mathcal{P}(E)$; i.e., $A \in \mathcal{C} \Leftrightarrow A+h \in \mathcal{C}$, for $h \in E$ (for example, the usual 4- and 8-connectivities are translation invariant). Given a structuring element B such that $B \in \mathcal{C}$ and B contains the origin of E , it can be shown [28] (see also [1, 11]) that $\mathbf{C} : \Sigma_0 \rightarrow \mathcal{P}(\mathcal{P}(E))$, given by

$$\mathbf{C}(\sigma) = \{A \in \mathcal{P}(E) \mid A \oplus |\sigma|B \in \mathcal{C}\}, \quad (21)$$

for $\sigma = -(R-1), \dots, 0$, is a connectivity pyramid. Here, $A \oplus B = \bigcup_{h \in B} (A+h)$ is the translation-invariant dilation of A by B [23], and $|\sigma|B$ is the dilation of B by itself repeated $|\sigma| - 1$ times (by convention, $0B$ is the origin). In this case, the top-level σ -connectivity class $\mathbf{C}(0)$ corresponds to \mathcal{C} , and the σ -connectivity classes below get increasingly larger as the radius $|s|$ increases. The connectivity measure $\varphi : \mathcal{P}(E) \rightarrow \Sigma$ in this case is given by

$$\varphi(A) = \max\{\sigma \in \Sigma_0 \mid A \oplus |\sigma|B \in \mathcal{C}\}, \quad A \subseteq E. \quad (22)$$

The value of the connectivity measure $\varphi(A)$ gives the extent to which A is disconnected with respect to the base connectivity \mathcal{C} . If $A \in \mathcal{C}$, then $\varphi(A)$ assumes its greatest value, $\varphi(A) = 0$, and A is fully connected. Otherwise, the degree of connectivity is strictly negative; the more negative the connectivity measure is, the more disconnected A is, in the sense that a larger dilation needs to be applied on A

in order to “reconnect” it. Figure 2 illustrates this case, where $E = \mathbb{R}^2$ and B is a disk structuring element centered at the origin.

The σ -grains of $A \in \mathcal{P}(E)$ according to the dilation-based multiscale connectivity discussed above are referred to as the σ -clusters of A . If C is a σ -cluster, then $C \oplus |\sigma|B \in \mathcal{C}$. The σ -clusters of A can be computed via the σ -connectivity openings [28]:

$$\gamma_x^\sigma(A) = A \cap \gamma_x(A \oplus |\sigma|B), \quad \sigma \in \Sigma_0, x \in A, \quad (23)$$

with $\gamma_x^\sigma(A) = \emptyset$ if $x \notin A$. Here, γ_x is the connectivity opening associated with the base connectivity class \mathcal{C} . From the above expression, it is easy to see that the dual σ -connectivity closings are given by (assuming a symmetric structuring element B):

$$\varphi_x^\sigma(A) = A \cup \varphi_x(A \ominus |\sigma|B), \quad \sigma \in \Sigma_0, x \notin A, \quad (24)$$

with $\varphi_x^\sigma(A) = E$ if $x \in A$. Here, φ_x is the connectivity closing associated with the base connectivity class \mathcal{C} , and $A \ominus B = \{h \in E \mid (B + h) \subseteq A\}$ is the translation-invariant erosion of A by B [23].

As for the σ -reconstruction operators, it can be shown [28] that

$$\rho^\sigma(A \mid M) = A \cap \rho(A \oplus |\sigma|B \mid A \cap M), \quad \sigma \in \Sigma_0, A, M \in \mathcal{P}(E), \quad (25)$$

where ρ is the reconstruction operator associated with the base connectivity class \mathcal{C} . It follows that the dual σ -reconstruction operators are given by:

$$\xi^\sigma(A \mid M) = A \cup \xi(A \ominus |\sigma|B \mid A \cup M), \quad \sigma \in \Sigma_0, A, M \in \mathcal{P}(E), \quad (26)$$

where ξ is the dual reconstruction operator associated with the base connectivity class \mathcal{C} .

3 Multiscale Connected Operators

In this section, we define connected operators in the context of multiscale connectivity, and investigate some of their properties.

3.1 Definition. Let (φ, \mathbf{C}) be a multiscale connectivity system on $\mathcal{P}(E)$. An operator ψ on T^E is said to be σ -connected if it is connected according to $\mathbf{C}(\sigma)$, for $\sigma \in \Sigma_0$.

A σ -connected operator is said to be connected at scale σ . An operator is said to be *fully connected* if it is connected at all scales, whereas it is said to be *fully disconnected* if it is not connected at any scale.

The following result is a direct consequence of Proposition 2.3.

3.2 Proposition. If ψ is a σ -connected operator, then ψ is τ -connected, for $\tau \geq \sigma$.

The above proposition says that if an operator is connected at a certain scale, it is connected at all larger scales. In other words, increasing the connectivity scale preserves the connectedness of the operator. However, nothing is guaranteed if one picks a smaller scale or one that is not comparable to the current one.

Motivated by the previous discussion, we define the notion of critical scale for a connected operator.

3.3 Definition. The *critical scale* of an operator ψ with respect to a given multiscale connectivity is defined by

$$\sigma^\psi = \bigwedge \{ \sigma \in \Sigma_0 \mid \psi \text{ is } \sigma\text{-connected} \}. \quad (27)$$

The notion of critical scale is most useful when the set of scales Σ is a chain, i.e., a totally ordered lattice. In this case, the critical scale divides Σ in two subsets: ψ is connected at all scales larger than σ^ψ , and not connected at all other scales, smaller than σ^ψ . If the scale set is a discrete chain, then ψ must of course be connected at the critical scale itself. However, this need not be the case in general. For a simple counter-example, consider a connectivity pyramid \mathbf{C} defined on a real scale set, such that $\mathbf{C}(\sigma) = \mathcal{C}$, for $\sigma > \sigma_0$, and $\mathbf{C}(\sigma) = \mathcal{C}' \supseteq \mathcal{C}$, for $\sigma \leq \sigma_0$. If ψ is connected with respect to \mathcal{C} but not to \mathcal{C}' , then $\sigma^\psi = \sigma_0$, even though ψ is not σ_0 -connected. An operator with a small critical scale tends to be “coarse” in nature, since it is a connected operator at scales for which the flat zones are larger. Conversely, a large critical scale may indicate a “finer” operator, since it fails to treat uniformly the larger flat zones associated with low connectivity scales.

As a simple example, consider a binary “close-holes” operator ψ , which fills connected regions of the background that are disconnected from the image domain boundary. Let us assume 8-connectivity for the image foreground and 4-connectivity for the image background, a common assumption that is made to deal with the “connectivity paradox” of digital images [29]. Consider the simple multiscale connectivity system discussed in Section 2.3, whereby $\mathbf{C} : \{1, 2\} \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{Z}^2))$ is given by $\mathbf{C}(1) = \{8\text{-connected subsets of } \mathbb{Z}^2\}$ and $\mathbf{C}(2) = \{4\text{-connected subsets of } \mathbb{Z}^2\}$. In this case, ψ is 2-connected, but not 1-connected, since the situation depicted in Figure 3 can occur; in this example, the close-holes operator has broken up the single flat zone (at scale 1) corresponding to the background. The critical scale here is the top scale, $\sigma^\psi = 2$. By contrast, if one assumes 4-connectivity for the foreground, and 8-connectivity for the background, then the corresponding close-holes operator would both 2- and 1-connected, with a smaller critical scale.

If we consider now a dilation-based multiscale connectivity (see Section 2.3), we notice that most ordinary connected operators defined in terms of the base connectivity class \mathcal{C} will be connected only at the top scale. For an operator to be connected at smaller scales (and thus to have a smaller critical scale), this operator must be, to some extent, “insensitive” to the clustering of flat zones introduced by the dilation; i.e., it must treat clusters of grains or pores uniformly (either keep them or remove them). The smaller the critical scale is, the larger the clusters of grains or pores that the operator treats uniformly are.

The following result, which deals with combining multiscale connected operators, follows directly from Propositions 2.4 and 3.2.

3.4 Proposition. Let ψ , ϕ , and $\{\psi_i \mid i \in I\}$ be operators on T^E .

- (a) If ψ is σ -connected and ϕ is τ -connected, then the composition $\psi\phi$ is $(\sigma \vee \tau)$ -connected.
- (b) If ψ_i is σ_i -connected, for $i \in I$, then the supremum $\bigvee \psi_i$ and the infimum $\bigwedge \psi_i$ are $(\bigvee \sigma_i)$ -connected. □

We now turn to *multiscale grain operators*. Given a multiscale connectivity system (φ, \mathbf{C}) on $\mathcal{P}(E)$, we define *foreground σ -grain operators* and *background σ -grain operators* by

$$\psi_{\sigma,u} = \bigvee_{x \in E} \iota_u \gamma_{\sigma,x}, \quad (28)$$

$$\phi_{\sigma,v} = \bigwedge_{x \in E} \kappa_v \varphi_{\sigma,x}. \quad (29)$$

Here, $\{\gamma_x^\sigma \mid \sigma \in \Sigma_0, x \in E\}$ and $\{\varphi_x^\sigma \mid \sigma \in \Sigma_0, x \in E\}$ are the σ -connectivity openings and σ -connectivity closings associated with (φ, \mathbf{C}) , and ι_u, κ_v are the foreground and background trivial operators, defined by (5) and (6), respectively. Note that $\psi_{\sigma,u}$ and $\phi_{\sigma,v}$ are σ -connected operators. In addition, $\psi_{\sigma,u}^* = \phi_{\sigma,u}$; i.e., the foreground σ -grain operator and the background σ -grain operator are dual to each other.

If u is increasing, then $\alpha_{\sigma,u} = \psi_{\sigma,u}$ is an opening on $\mathcal{P}(E)$, called a *σ -grain opening*. Similarly, if v is increasing, then $\beta_{\sigma,v} = \phi_{\sigma,v}$ is a closing on $\mathcal{P}(E)$, called a *σ -grain closing*. For example, the σ -connectivity opening γ_x^σ and the σ -reconstruction operator $\rho^\sigma(\cdot \mid M)$ are σ -grain openings, whereas the σ -connectivity closing φ_x^σ and the dual σ -reconstruction operator $\xi^\sigma(\cdot \mid M)$ are σ -grain closings.

We have the following result.

3.5 Proposition. For given increasing criteria u and v ,

- (a) The family of σ -grain openings $\{\alpha_{\sigma,u} \mid \sigma \in \Sigma_0\}$ is a granulometry; i.e, $\alpha_{\sigma,u} \leq \alpha_{\tau,u}$, for $\sigma \geq \tau$.

(b) The family of σ -grain closings $\{\beta_{\sigma,u} \mid \sigma \in \Sigma_0\}$ is an anti-granulometry; i.e, $\beta_{\sigma,u} \geq \beta_{\tau,u}$, for $\sigma \geq \tau$.

PROOF. (a): If $\sigma \geq \tau$, we have that $\gamma_{\sigma,x} \leq \gamma_{\tau,x} \Rightarrow \iota_u \gamma_{\sigma,x} \leq \iota_u \gamma_{\tau,x} \Rightarrow \alpha_{\sigma,u} = \bigvee \iota_u \gamma_{\sigma,x} \leq \bigvee \iota_u \gamma_{\tau,x} = \alpha_{\tau,u}$, as required.

(b): The proof is analogous. Q.E.D.

Usual granulometries and anti-granulometries based on grain openings and grain closings are indexed by the criteria u and v . On the other hand, the granulometries and anti-granulometries defined in Proposition 3.5 are indexed by the scale parameter, with fixed criteria u and v . This leads therefore to a novel type of granulometric analysis. The notion of scale associated with usual granulometries (the criteria u and v) are typically related to *size* [22], whereas here the notion of scale (the parameter s) has to do with connectivity. Both kinds of information can be used in granulometric analysis of images (see Section 4).

If the criterion u or v is not increasing, then the family $\{\alpha_{\sigma,u} \mid \sigma \in \Sigma_0\}$ or $\{\beta_{\sigma,v} \mid \sigma \in \Sigma_0\}$ does not have regular ordering with respect to the scale parameter σ . However, one can define the following operators:

$$v_{\sigma,u} = \bigwedge_{\tau \leq \sigma} \psi_{\tau,u} \quad (30)$$

$$\omega_{\sigma,v} = \bigvee_{\tau \leq \sigma} \phi_{\tau,v}. \quad (31)$$

From Proposition 3.4(b) it follows that these operators are σ -connected. Furthermore, they are dual to each other: $v_{\sigma,u}^* = \omega_{\sigma,v}$. Also note that, for $\sigma \geq \tau$, we have that $v_{\sigma,u} \leq v_{\tau,u}$ and $\omega_{\sigma,v} \geq \omega_{\tau,v}$. These properties resemble those of σ -grain openings and σ -grain closings. As a matter of fact, if u is increasing, then $v_{\sigma,u} = \psi_{\sigma,u} = \alpha_{\sigma,u}$; similarly, if v is increasing, then $\omega_{\sigma,v} = \phi_{\sigma,v} = \beta_{\sigma,v}$.

Similarly to the single-scale case, we can define a class of operators that simultaneously act on the foreground and the background:

$$\begin{aligned} \zeta_{\sigma,u,v}(A) &= \bigcup \{C \mid C \leq_{\sigma} A \text{ and } u(C) = 1 \text{ or } C \leq_{\sigma} A^c \text{ and } v(C) = 0\} \\ &= \psi_{\sigma,u}(A) \cup [A^c \cap \phi_{\sigma,v}(A)] = [\psi_{\sigma,u}(A) \cup A^c] \cap \phi_{\sigma,v}(A), \quad A \in \mathcal{P}(E). \end{aligned} \quad (32)$$

These are called σ -grain operators or *set σ -flattenings* and generalize foreground and background σ -grain operators, since $\psi_{\sigma,u} = \zeta_{\sigma,u,1}$ and $\phi_{\sigma,v} = \zeta_{\sigma,1,v}$. A stable σ -grain operator is a *set σ -leveling*.

Lowering the connectivity scale may destroy the property of connectedness of an operator. One might wonder what happens as regards to the property of locality (i.e., the property of being a grain

operator). The following result shows that locality is preserved as long as connectedness is (and vice-versa).

3.6 Proposition. Let ψ be a σ -grain operator on $\mathcal{P}(E)$. For $\tau \leq \sigma$,

$$\psi \text{ is a } \tau\text{-connected operator} \Leftrightarrow \psi \text{ is a } \tau\text{-grain operator.} \quad (33)$$

PROOF. “ \Rightarrow ”: Let $A_1, A_2 \in \mathcal{P}(E)$, and let C be a flat σ -zone (a σ -grain or σ -pore) of both A_1 and A_2 . Since, by hypothesis, ψ is τ -connected, it follows from Proposition 2.7 that it suffices to show that $\psi(A_1)(C) = \psi(A_2)(C)$. We have that $C = \bigcup C_\alpha$, where C_α are flat σ -zones of A_1 and A_2 . But, since ψ is a σ -grain operator, we have that $\psi(A_1)(C_\alpha) = \psi(A_2)(C_\alpha)$, for each α . Hence $\psi(A_1)(C) = \psi(A_2)(C)$, as required.

“ \Leftarrow ”: This implication is obvious. Q.E.D.

Hence, lowering the connectivity scale does not destroy the locality of a an operator as long as the operator remains connected. This does not work in the reverse direction: increasing the connectivity scale may destroy locality, even though it preserves connectedness (as is remarked in [8], there is no counterpart of Proposition 2.3 for grain operators).

4 Applications to Image Analysis

In this Section, we demonstrate the application of multiscale connected operators in image analysis tasks such as scale-space representation, granulometric analysis and automatic target recognition.

4.1 Scale-Space Representation

In many image analysis applications, it is of interest to track the evolution of image structures (e.g., regional maxima and minima) through a continuous range of scales, from fine to coarse. Such a representation is known in the literature as a *scale-space* [30, 31].

Mathematical Morphology operators are inherently suitable for scale-space representations. In this section, we demonstrate the use of *grayscale σ -levelings*, derived from the binary σ -levelings discussed previously, in scale-space representation. In [32], the use of grayscale levelings as a tool for scale-space representation is discussed. The main novelty in the approach described here is that the role of scale is played by the connectivity scale associated with a multiscale connectivity.

Let T be an interval of integer numbers, so that T^E consists of discrete-valued grayscale images. Consider a family $\{\psi_t \mid t \in T\}$ of increasing operators on $\mathcal{P}(E)$, which is decreasing with respect to t

(i.e., $s \leq t \Rightarrow \psi_t \leq \psi_s$). We define an operator $\tilde{\psi}$ on T^E by:

$$\tilde{\psi}(f)(v) = \bigvee \{t \in T \mid v \in \psi_t(X_t(f))\}, \quad v \in E, \quad (34)$$

where $X_t(f) = \{v \in E \mid f(v) \geq t\}$. The operator $\tilde{\psi}$ is referred to as the *semi-flat grayscale operator* generated by the family $\{\psi_t \mid t \in T\}$ [23]. It can be easily seen that $\tilde{\psi}$ is an increasing operator. In the particular case when $\psi_t = \psi$, for every $t \in T$ (i.e., the family consists of a single operator ψ), (34) reduces to

$$\bar{\psi}(f)(v) = \bigvee \{t \in T \mid v \in \psi(X_t(f))\}, \quad v \in E. \quad (35)$$

This is known as the *flat grayscale operator* generated by ψ [23]. It can be shown that the semi-flat and flat operators generated by dilations, erosions, openings, closings on $\mathcal{P}(E)$ are dilations, erosions, openings and closings on T^E , respectively [23]. In addition, it can be shown that the semi-flat and flat grayscale operators generated by increasing connected operators on $\mathcal{P}(E)$ are (increasing) connected operators on T^E (see [4]).

Given a connectivity class in $\mathcal{P}(E)$, and the associated reconstruction operator $\rho: \mathcal{P}(E) \times \mathcal{P}(E) \rightarrow \mathcal{P}(E)$, we define the *grayscale reconstruction* $\tilde{\rho}: T^E \times T^E \rightarrow T^E$ by

$$\tilde{\rho}(f \mid g)(v) = \bigvee \{t \in T \mid v \in \rho(X_t(f) \mid X_t(g))\}, \quad v \in E, \quad (36)$$

Note that $\tilde{\rho}(\cdot \mid g)$ is the semi-flat operator generated by the family of binary operators $\{\psi_t(\cdot) = \rho(\cdot \mid X_t(g)) \mid t \in T\}$ — the operator is semi-flat, in the sense of [23], as each ψ_t is a distinct operator on $\mathcal{P}(E)$. In an analogous way, one can define the *grayscale dual reconstruction* operator $\tilde{\xi}(f \mid g)$. From our previous discussion, it follows that $\tilde{\rho}(\cdot \mid g)$ is a connected opening in T^E , whereas $\tilde{\xi}(\cdot \mid g)$ is a connected closing in T^E . The grayscale reconstruction operator and its dual are very useful in image analysis applications [26].

A *grayscale flattening* is defined in [9] as a grayscale connected operator such that any transition between adjacent pixels in the transformed image is bracketed by a larger transition in the original image. A *grayscale leveling* is defined as a grayscale flattening that in addition preserves the orientation of the transitions. The importance of grayscale levelings in scale-space representation comes from the fact that they do not create new regional maxima or minima as the image evolves through scale-space [9, 32]. It can be shown by using the results in [9] that flat and semi-flat grayscale operators generated by binary connected operators are grayscale flattenings. If the generating binary connected operators are in addition stable, then the resulting grayscale operators are levelings.

Now, consider binary σ -grain openings and binary σ -grain closings, associated with a given multiscale connectivity on $\mathcal{P}(E)$ (see Section 3). These are stable connected operators; hence, their flat or semi-flat extensions are grayscale levelings, which we refer to as *grayscale σ -levelings*.

Let us consider in particular the family of σ -reconstruction operators $\{\rho^\sigma(\cdot|M) \mid \sigma \in \Sigma_0\}$ and dual σ -reconstruction operators $\{\xi^\sigma(\cdot|M) \mid \sigma \in \Sigma_0\}$. Their semi-flat extensions are respectively the family of grayscale σ -reconstruction operators $\{\tilde{\rho}^\sigma(\cdot|g) \mid \sigma \in \Sigma_0\}$ and grayscale dual σ -reconstruction operators $\{\tilde{\xi}^\sigma(\cdot|g) \mid \sigma \in \Sigma_0\}$ on T^E . Clearly, these are respectively granulometries and anti-granulometries of grayscale σ -levelings, and are suitable to be used in scale-space representation of grayscale images, where the role of scale is played by the connectivity scale associated with a multiscale connectivity.

For a specific example, let us consider the dilation-based multiscale connectivity discussed in Section 2.3. It follows from (25) that the associated grayscale σ -reconstruction operator is given by:

$$\tilde{\rho}^\sigma(f|g) = f \wedge \tilde{\rho}\left(f \oplus |\sigma|B \mid f \wedge g\right), \quad \sigma \in \Sigma_0, f, g \in T^E, \quad (37)$$

where “ \wedge ” is the infimum operation for grayscale images, $f \oplus |\sigma|B$ is the flat dilation of f by $|\sigma|B$ (which is defined as the flat extension of the usual translation-invariant binary dilation), and $\tilde{\rho}$ is the grayscale reconstruction operator obtained from the base connectivity. Similarly, it follows from (26) that the grayscale dual σ -reconstruction operator associated with a dilation-based multiscale connectivity is given by (assuming a symmetric structuring element B):

$$\tilde{\xi}^\sigma(f|g) = f \vee \tilde{\xi}\left(f \ominus |\sigma|B \mid f \vee g\right), \quad \sigma \in \Sigma_0, f, g \in T^E, \quad (38)$$

where “ \vee ” is the infimum operation for grayscale images, $f \ominus |\sigma|B$ is the flat erosion of f by $|\sigma|B$ (which is defined as the flat extension of the usual translation-invariant binary erosion), and $\tilde{\xi}$ is the grayscale dual reconstruction operator obtained from the base connectivity.

Assuming that the base connectivity is one of the usual digital connectivities, there are efficient implementations available for both $\tilde{\rho}$ and $\tilde{\xi}$ (see [26]). Equations (37) and (38) provide therefore practical algorithms to compute the associated scale-spaces.

Fig. 4 illustrates a scale-space representation using both the σ -reconstruction and dual σ -reconstruction operators given in (37) and (38), for the classic cameraman image. Here, the structuring element B that generates the dilation-based multiscale connectivity is a 3×3 cross centered at the origin, and the base connectivity class is given by 4-connectivity. Fig 4(a) depicts the original image, the marker used for the σ -reconstructive scale-space (obtained by a structural opening by a city-block disk of radius 3), and the marker used for the dual σ -reconstructive scale-space (obtained by a structural closing by a city-block

disk of radius 5). Fig 4(b) and Fig. 4(c) depict the evolution, from smaller to larger connectivity scale, for the anti-extensive σ -reconstructive and the extensive dual σ -reconstructive scale-spaces, respectively. The former scale-space evolves towards the suppression of bright features, while the latter scale-space evolves towards the suppression of dark features. Note that in both cases the isolated features are the ones that are suppressed first (for example, the bright and dark structures at the top of the tallest building, or the cameraman's bright shirt and dark eye details).

4.2 Granulometric Analysis

Let (φ, \mathbf{C}) be a multiscale connectivity system on $\mathcal{P}(E)$. Given an increasing criterion $u : \mathcal{P}(E) \rightarrow \{0, 1\}$, consider the granulometry of σ -grain openings $\{\alpha_{\sigma, u} \mid \sigma \in \Sigma_0\}$ discussed in Section 3.

Consider also a family of increasing criteria $\{u_t \mid t \in T\}$, where T is a poset, such that $u_{t_1} \leq u_{t_2}$ if $t_1 \geq t_2$. It is easy to verify that, given a fixed scale $\sigma \in \Sigma_0$, the family of σ -grain openings $\{\alpha_{\sigma, u_t} \mid t \in T\}$ is a granulometry. The criteria u_t are typically associated with *size*; e.g., it is common to define $u_t(A) = 1$, if $S(A) \geq t$, where S is some set function measuring area, volume, etc. (we deliberately leave aside measure-theoretic issues here.)

By combining these two kinds of granulometries of connected openings, one can achieve a description of an image that contains multiscale information both from size and connectivity. From this point on, let E be a bounded domain in \mathbb{Z}^2 , let Σ and T be intervals of integer numbers, and let $S : \mathcal{P}(E) \rightarrow \mathbb{Z}$ be the set cardinality function (thus, S measures the area of discrete binary images). Given $A \in \mathcal{P}(E)$, define the function $C_A : \Sigma_0 \times T \rightarrow \mathbb{Z}$ by

$$C_A(\sigma, t) = S(\alpha_{\sigma, u_t}(A)), \quad \sigma \in \Sigma_0, t \in T, \quad (39)$$

where u_t is defined by $u_t(A) = 1$, if $S(A) \geq t$. The function C_A , which we call a *size-connectivity distribution*, contains joint multiscale information about size and connectivity content of image A — this is a bivariate granulometric distribution, in the sense of [33]. Note that C_A is a decreasing function in both arguments σ and t .

By taking discrete derivatives of C_A , one obtain families of *pattern spectra* [22] for size and connectivity. By ranging over the connectivity scale σ , one defines a family of size pattern spectra $\{P_{A, \sigma}^{\text{SIZ}} \mid \sigma \in \Sigma_0\}$, given by:

$$P_{A, \sigma}^{\text{SIZ}}(t) = -\frac{\partial}{\partial t} C_A(\sigma, t), \quad t \in T, \quad (40)$$

for each fixed $\sigma \in \Sigma_0$. Conversely, by ranging over the size t , one defines a family of connectivity pattern

spectra $\{P_{A,t}^{\text{CON}} \mid t \in T\}$, given by:

$$P_{A,t}^{\text{CON}}(\sigma) = -\frac{\partial}{\partial \sigma} C_A(\sigma, t), \quad \sigma \in \Sigma_0, \quad (41)$$

for each fixed $t \in T$.

For a concrete example, consider the grayscale image of circular nodules, depicted in Fig. 5(a). Binarization by means of simple thresholding results in the image shown in Fig. 5(b). Consider a dilation-based multiscale connectivity, where $\Sigma_0 = \{-12, -11, \dots, -1, 0\}$ (the value O_Σ is not relevant here), the structuring element B that generates the multiscale connectivity is a 3×3 cross centered at the origin, and the base connectivity class is given by 4-connectivity. In addition, let the size criterion be the area (number of pixels), with $T = \{100, 200, \dots, 1000\}$. The action of the connected σ -grain opening $\alpha_{\sigma, ut}$ for a few values of σ and t is shown in Fig. 5(c). The size-connectivity distribution of this image is depicted in Fig. 5(d) as a 2-D mesh, while the corresponding size and connectivity pattern spectra are displayed by means of “waterfall” plots in Fig. 5(d) and (e).

4.3 Automatic Target Recognition

We will now demonstrate a multiscale motion-based connected operator, designed as part of an algorithm for automatic target detection and tracking in FLIR (forward-looking infrared) image sequences [34]. This operator turns out to be a multiscale σ -grain opening.

The original images here consist of video sequences acquired from an airborne infrared camera, provided by the *U.S. Army Missile Command* (MICOM). Fig. 6 shows the first five frames from one of the available image sequences. Note that the targets appear as bright features.

After an intraframe detection step, one obtains a sequence of binary images, which contain targets, as well as false alarms generated by clutter. In order to reduce the false alarms, an interframe processing step based on a connected σ -grain opening is used. This step imposes a motion criterion that reflects the fact that the targets of interest display a continuous trajectory across frames and have limited relative motion with respect to the FLIR sensor. The underlying connectivity needs to take into account temporal undersampling and sensor jitter, which makes the targets “jump” from frame to frame. The solution is to use a dilation-based multiscale connectivity, where the base connectivity class is given by 3-D digital 6-adjacency connectivity (connectivity across the faces of each voxel), and the 3-D dilation uses as structuring element a purely spatial (2-D) 3×3 cross centered at the origin. The criterion u here is defined as follows: $u(A) = 1$ if A spans at least three consecutive frames. So, candidate targets that do not display persistence are discarded (these are likely to be false alarms). The smaller the

scale is, i.e., the larger the dilation that is applied, the more “jumpy” the trajectories are allowed to be (however, this also increases the false alarm rate). The action of the resulting motion-based connected σ -grain opening $\alpha_{\sigma,u}$ is illustrated in Fig. 7, at a fixed connectivity scale σ . The objects removed by the motion-based connected operator correspond mostly to either residual clutter from the intraframe processing step, or targets that move too fast with respect to the sensor, which are not considered to be the primary targets of interest. As is demonstrated in [34], the results achieved by using this approach are very good.

In [34], a fixed connectivity scale, empirically determined, was employed. Here, we examine the multiscale properties of this algorithm by means of the granulometry $\{\alpha_{\sigma,u} \mid \sigma \in \Sigma_0\}$, where $\Sigma_0 = \{-16, -15, \dots, -1, 0\}$ (as before, the value of O_Σ does not matter here), and u is as defined above. The image sequence used is the same depicted in Fig. 6. This sequence contains a total of 27 frames, and the processed initial binary sequence contains an average of 3.2 candidate targets per frame. The curve displayed in Fig. 8 plots the average number of tracked targets per frame after application of the motion-based connected operator, versus the connectivity scale. Moving to the left on this graph, i.e., towards smaller connectivity scales, reduces the number of targets that are removed by the motion-based connected operator; this has the effect of reducing the false negative rate, at the expense of increasing the false positive (false alarm) rate. Moving towards larger connectivity scales has the opposite effect. Changing the connectivity scale in this case has the effect of moving the operating point of the algorithm, i.e., the balance between false negatives and false positives.

5 Conclusions

We believe that multiscale connected operators can offer a new interesting theoretical perspective on connected operators, as well as, from an applications point of view, add a new tool to the image analysis practitioner’s repertoire.

We have considered here extensive and anti-extensive grayscale σ -levelings. The algorithms to compute self-dual grayscale levelings are not trivial, and are based on arc-wise connectivities. The study of such algorithms to compute self-dual σ -levelings for arbitrary multiscale connectivities is an interesting problem.

In addition, we have addressed in this paper only “flat” connected operators, that is, the classical type of such operators. In recent publications, F. Meyer and J. Serra have proposed the notion of “extended” connected operators, where the zones of the partition are allowed to be non-flat (typically

observing a Lipschitz-type condition). The generalization of the work presented here to this case is a point for future research.

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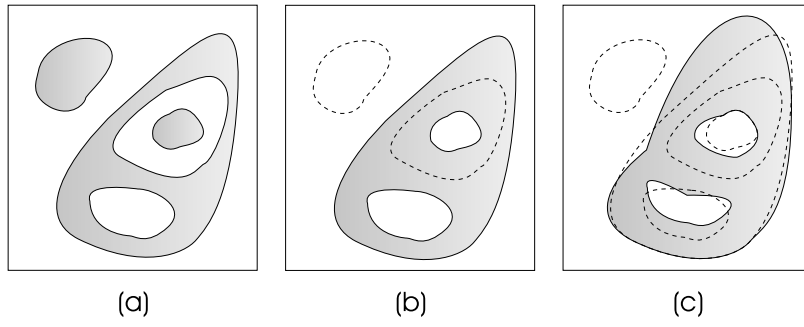


Figure 1: (a) A binary image A . (b) The output of a connected operator applied on A . (c) This image cannot possibly be the output of a connected operator applied on A .

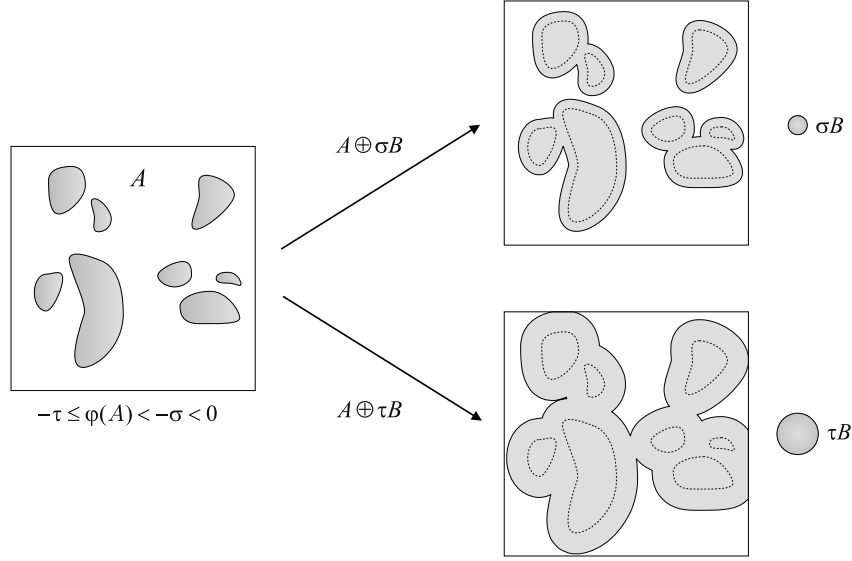


Figure 2: Illustration of dilation-pyramid multiscale connectivity. The original set A is disconnected; hence its connectivity measure $\varphi(A)$ is negative. Note that $A \oplus \tau B$ is connected, but $A \oplus \sigma B$ is not. Therefore, A is $-\tau$ -connected but not $-\sigma$ -connected. Equivalently, $-\tau \leq \varphi(A) < -\sigma$. Note that A is composed of eight 0-clusters, four $-\sigma$ -clusters, but only one $-\tau$ -cluster.

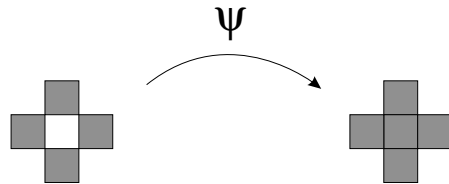


Figure 3: The close-holes operator ψ breaks up the single 8-connected flat zone corresponding to the background.



Figure 4: Illustration of scale-space representation. Top row: original grayscale image and markers. Middle row: evolution of the anti-extensive σ -reconstructive scale-space. Bottom row: evolution of the extensive dual σ -reconstructive scale-space.

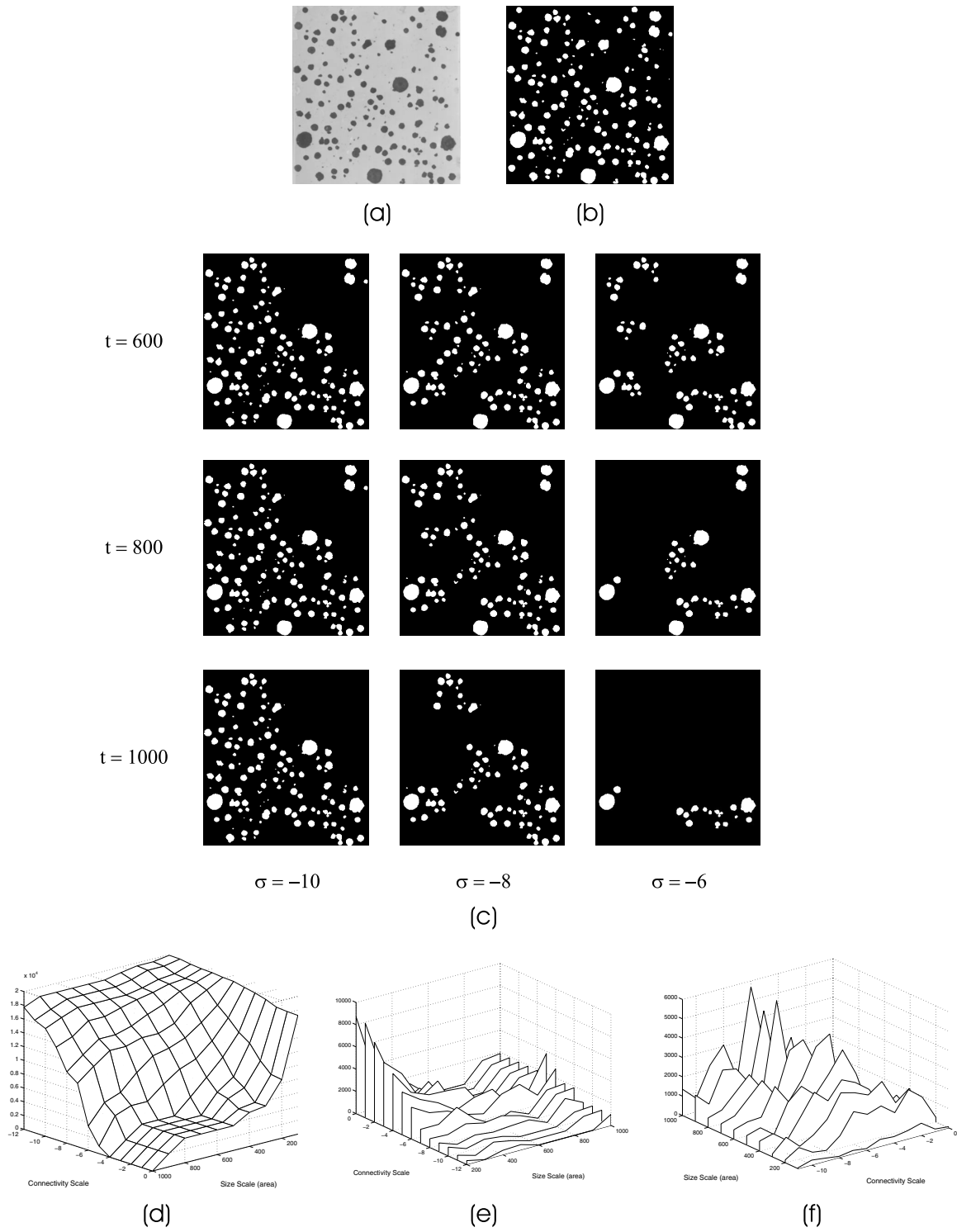


Figure 5: Illustration of granulometric analysis. (a) Original grayscale image. (b) Binarization. (c) The output of the connected σ -grain opening α_{σ, u_t} for a few values of σ and t . (d) Size-connectivity distribution. (e) Size pattern spectra. (f) Connectivity pattern spectra.

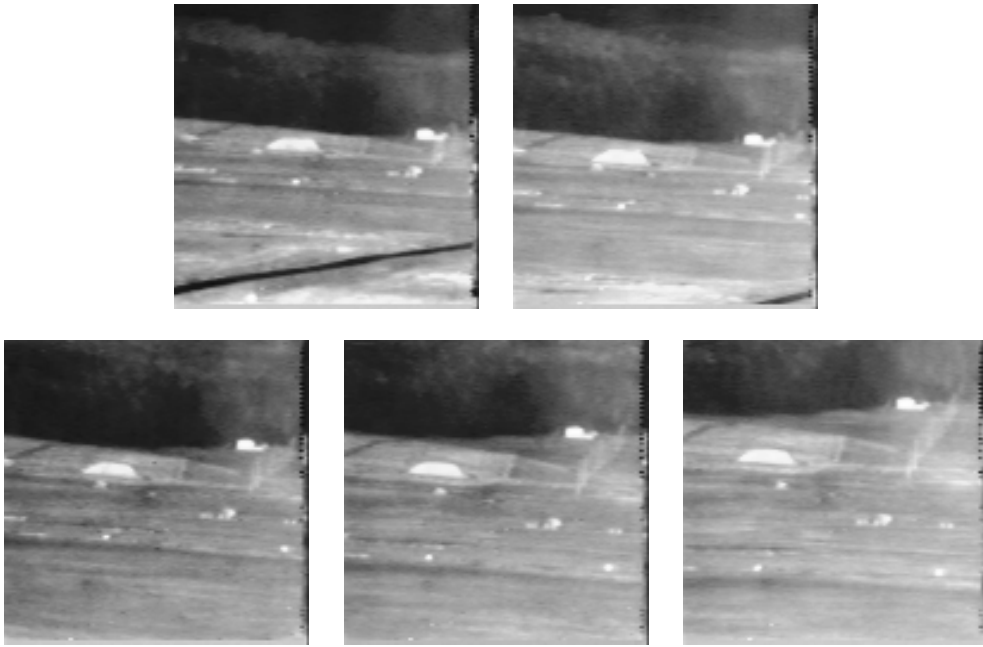


Figure 6: First five frames of a sample FLIR image sequence.

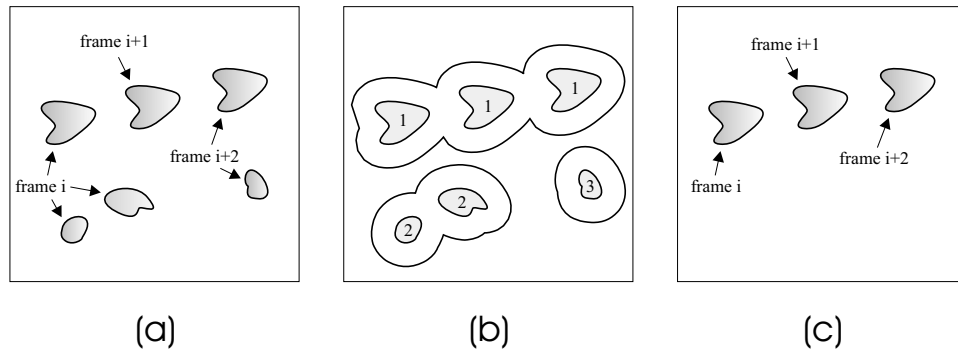


Figure 7: Action of motion-based connected operator, at a fixed connectivity scale σ . (a) Objects in three consecutive frames. (b) Spatio-temporal σ -connected components, according to the dilation-pyramid multiscale connectivity employed. (c) Output of the motion-based connected operator, which eliminates spatio-temporal σ -connected components that do not span at least three consecutive frames.

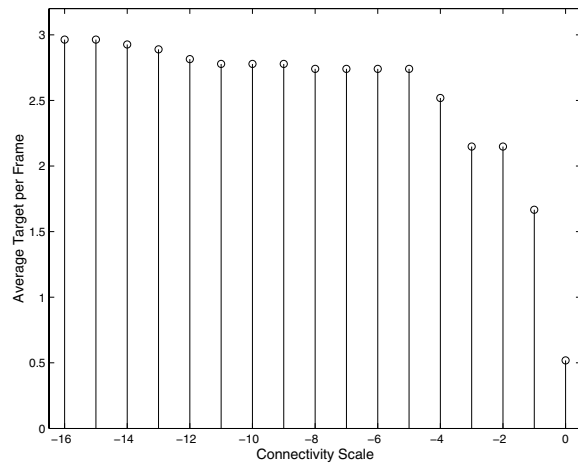


Figure 8: Average target per frame after application of the motion-based connected operator versus the connectivity scale.